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Mesons in marginally deformed AdS/CFT

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Abstract: We study the embedding of spacetime filling D7-branes in β-deformed backgrounds which, according to the AdS/CFT dictionary, corresponds to flavoring β-deformed $\mathcal{N} = 4$ super Yang-Mills. We consider supersymmetric and more general non-supersymmetric three parameter deformations. The equations of motion for quadratic fluctuations of a probe D7-brane wrapped on a deformed three-sphere exhibit a non-trivial coupling between scalar and vector modes induced by the deformation. Nevertheless, we manage to solve them analytically and find that the mesonic mass spectrum is discrete, with a mass gap and a Zeeman-like splitting occurs. Finally we propose the action for the dual field theory as obtained by $*$-product deformation of super Yang-Mills with fundamental matter.

Keywords: AdS-CFT Correspondence, D-branes, Gauge-gravity correspondence.

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1. Introduction

One of the main challenges of the elementary particle theoretical physics is the understanding of the low energy regime of confining theories, primarily QCD. Progress in this direction is expected in the context of AdS/CFT correspondence [1] which allows for a dual description of Yang-Mills theories at strong coupling in terms of a perturbative string/supergravity theory.

In this respect, a quite recent progress concerns the generalization of the AdS/CFT correspondence to include matter in the fundamental representation of the gauge group [2, 3]. The holographic description of a 4D supersymmetric Yang-Mills theory with fundamental matter can be obtained by considering a system of intersecting D3-D7 branes. Precisely, the near horizon geometry of a system of N D3-branes in the presence of N_f spacetime-filling D7-branes, in the large N limit and N_f fixed, gives the dual description of a $\mathcal{N} = 4$ SU(N) SYM theory living on the D3-branes with supersymmetry broken to $\mathcal{N} = 2$ by N_f hypermultiplets in the fundamental representation of SU(N). The field content of the

hypermultiplets is given by excitations of fundamental strings stretching between D3 and D7-branes.

When the D3 and the D7-branes are separated along the mutual orthogonal directions the hypermultiplets acquire a mass which is proportional to the distance between the branes. For coincident branes (vanishing masses) the $\mathcal{N} = 2$ theory is superconformal invariant.

As proposed in [3] (see also [4]), excitations of fundamental strings with both ends on the D7-branes represent mesonic states of the corresponding SYM field theory. Studying these fluctuations allows for determining the mass spectrum of the mesonic excitations. The spectrum turns out to be discrete with a mass gap [5].

Since the original proposal of inserting D7-branes in the standard $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry, a lot of work has been done in the direction of finding generalizations to less supersymmetric and/or non-conformal backgrounds. In particular, flavors and meson spectra on the conifold and in the Klebanov-Strassler model have been studied in [6]. The Maldacena-Nunez background has been considered in [7], the class of metrics of the form $\mathrm{AdS}_5 \times Y^{p,q}$ and $\mathrm{AdS}_5 \times L^{a,b,c}$ in [8], while for the Polchinski-Strassler set-up see [9]. Supersymmetric embeddings of D-branes and their fluctuations in non-commutative theories have been investigated in [10]. Further generalizations concern other stable brane systems [11, 12]. Chiral symmetry breaking and theories at finite temperature have been first studied in [13, 14]. Moreover, several attempts have been devoted to going beyond the probe approximation and studying full back-reacted (super)gravity solutions [15]. Further interesting results can be found in [16 – 20].

Among the formulations of the AdS/CFT correspondence with less supersymmetry, the one-parameter Lunin-Maldacena (LM) background [21] corresponding to $\mathcal{N} = 1$ β-deformed SYM theories plays an interesting role, being the field theory and the dual string geometry explicitly known. The gravitational background is $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^5$ where $\tilde{\mathrm{S}}^5$ is the β-deformed five sphere obtained by performing a TsT transformation on a 2-torus inside the S^5 of the original background. This operation breaks the SO(6) symmetry group of the five sphere down to $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$. On the field theory side, this deformation corresponds to promoting the ordinary products among the fields in the $\mathcal{N} = 4$ action to a $*$-product which depends on the charges of the fields under two $\mathrm{U}(1)$'s and allowing for the chiral coupling constant to be different from the gauge coupling. Consistently with what happens on the string side, these operations break $\mathcal{N} = 4$ to $\mathcal{N} = 1$ supersymmetry, as the third $\mathrm{U}(1)$ (the one not involved in the $*$-product) corresponds to the R-symmetry. Further generalizations [22] lead to a dual correspondence between a non-supersymmetric Yang-Mills theory and a deformed LM background depending on three different real parameters γ_1, γ_2 and γ_3.[1]

All these models are (super)conformal invariant since the string geometry still has an AdS factor. As such they cannot be used to give a realistic description of the RG flow of a gauge theory towards a confining phase. However, it is interesting to investigate what

[1] We use the standard convention to name *real* deformation parameters with γ.

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happens if we insert D7-branes in these deformed backgrounds.[2] In particular, we expect to find a parametric dependence of the mesonic spectrum on γ_i's which could then be used to fine-tune the results.

In what follows we accomplish this project by studying the effects of inserting D7-branes in the more general non-supersymmetric LM-Frolov background. In the probe approximation ($N_f \ll N$), we first study the stability of the D3-D7 configuration. We find that, independently of the value of the deformation parameters, an embedding can be found which is stable, BPS and in the $\gamma_1 = \gamma_2 = \gamma_3$ case it is also supersymmetric.

We then study fluctuations of a D7-brane around the static embedding which correspond to scalar and vector mesons of the dual field theory. We consider the equations of motion for the tower of Kaluza-Klein modes arising from the compactification of the D7-brane on a deformed three-sphere. The background deformation induces a non-trivial coupling between scalar and vector modes. However, with a suitable field redefinition, we manage to simplify the equations and solve them analytically, so determining the mass spectrum exactly.

The effects of the deformation on the mesonic mass spectrum and on the corresponding KK modes are the following: i) As in the undeformed case the mass spectrum is discrete and with a mass gap, but it acquires a non-trivial dependence on the deformation parameters. Precisely, it depends on the parameters γ_2, γ_3 which are associated to TsT transformations along the tori with a direction orthogonal to the probe branes, whereas the parameter γ_1 associated to the deformation along the torus inside the D7 worldvolume never enters the equations of motion for quadratic fluctuations and does not affect the mass spectrum. ii) Since the deformation breaks SO(4) (the isomorphisms of the three-sphere) to $\mathrm{U}(1)\times\mathrm{U}(1)$ a Zeeman-like effect occurs and the masses exhibit a non-trivial dependence on the (m_2, m_3) quantum numbers associated to the two U(1)'s. The dependence is through the linear combination $(\gamma_2 m_3 - \gamma_3 m_2)^2$ so that the mass eigenvalues are smoothly related to the ones of the undeformed case by sending $\gamma_i \to 0$. iii) The corresponding eigenstates are classified according to their SO(4) and $\mathrm{U}(1) \times \mathrm{U}(1)$ quantum numbers. Expanding in vector and scalar harmonics on the three-sphere, we find Type I elementary fluctuations[3] in the $(\frac{l\mp1}{2}, \frac{l\pm1}{2})_{(m_2,m_3)}$ representations and Type II, Type III and scalar modes in the $(\frac{l}{2}, \frac{l}{2})_{(m_2,m_3)}$. For a given l the total number of degrees of freedom is $8(l+1)^2$ as in the undeformed theory but, given the degeneracy breaking, they split among different eigenvalues. For any given triplet (l, m_2, m_3) we compute the degeneracy of the corresponding mass eigenvalue. We find that the splitting is different according to the choice $\gamma_2 \neq \gamma_3$ or $\gamma_2 = \gamma_3$ (which includes the $\mathcal{N} = 1$ supersymmetric deformation). In the last case the spectrum exhibits a mass degeneracy between scalars and vectors which is remnant of the $\mathcal{N} = 2$ supersymmetric, undeformed case.

The paper is organized as follows. In section 2 we review the three-parameter deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using a set of coordinates suitable for the introduction of D7-branes. In section 3 we study the static embedding of a D7-brane and discuss its sta-

[2] Several works in the literature are devoted to the study of D-branes in this context [23 – 29].

[3] We use the classification of [5].

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bility. In the $\gamma_1 = \gamma_2 = \gamma_3$ case, using the results of [28] we argue that our configuration is supersymmetric. We then find the equations of motion for the bosonic fluctuations of a D7-brane in section 4 and solve them analytically in section 5 determining the exact mass spectrum. In section 6 we discuss the properties of the spectrum and analyze in detail the splitting of the mass levels and the corresponding degeneracy. Finally, in section 7 we formulate the field theory dual to our configuration, whereas our conclusions, comments and perspectives are collected in section 8.

2. Generalities on the three-parameter deformation of $\mathbf{AdS_5 \times S^5}$

Following [21, 22] we consider a type IIB supergravity background obtained as a three-parameter deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$. It is realized by three TsT transformations (T duality — angle shift — T duality) along three tori inside S^5 and driven by three different real parameters γ_i. The corresponding metric is usually written in terms of radial/toroidal coordinates (ρ_i, ϕ_i), $i = 1, 2, 3$, $\sum_i \rho_i^2 = 1$ on the deformed sphere, and in string frame it reads (we set $\alpha' = 1$)

$$\begin{aligned} ds^2 &= \frac{u^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{u^2}du^2 + R^2\left[\sum_i (d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_i \hat{\gamma}_i d\phi_i\right)^2\right] \\ G^{-1} &= 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_2^2\rho_3^2\rho_1^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 \qquad\qquad \hat{\gamma}_i \equiv R^2\gamma_i \end{aligned} \tag{2.1}$$

where R is the AdS_5 and S^5 radius. A further change of coordinates may be useful (we use the notation $c_\xi \equiv \cos\xi, s_\xi \equiv \sin\xi$ for any angle ξ)

$$\rho_1 = c_\alpha \quad , \quad \rho_2 = s_\alpha c_\theta \quad , \quad \rho_3 = s_\alpha s_\theta \tag{2.2}$$

leading to the description of this background in terms of Minkowski coordinates x^μ plus the AdS_5 coordinate u and five angular coordinates $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$. The deformations correspond to TsT transformations along the three tori (ϕ_1, ϕ_2), (ϕ_1, ϕ_3), (ϕ_2, ϕ_3) and are parametrized by constants $\hat{\gamma}_3$, $\hat{\gamma}_2$ and $\hat{\gamma}_1$ respectively.

This background is non-supersymmetric and it is dual to a non-supersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM (the deformation has to be exactly marginal since the AdS factor is not affected by TsT's). The $\mathcal{N} = 1$ supersymmetric background of [21] can be recovered by setting $\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3$.

With the aim of embedding D7-branes in this background we find more convenient to express the metric in terms of a slightly different set of coordinates. We describe the six dimensional internal space in terms of $X^m \equiv \{\rho, \theta, \phi_2, \phi_3, X_5, X_6\}$ which are mapped into the previous set of coordinates by the change of variables

$$\rho = u\, s_\alpha \quad , \quad X_5 = u\, c_\alpha\, c_{\phi_1} \quad , \quad X_6 = u\, c_\alpha\, s_{\phi_1} \tag{2.3}$$

In string frame and still setting $\alpha' = 1$, we then have

$$ds^2 = \frac{u^2}{R^2}\,\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{u^2}\,G_{mn}dX^m dX^n \tag{2.4}$$

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where the non-vanishing components of the metric G_{mn} are

$$
\begin{aligned}
G_{\rho\rho} &= 1 & G_{\theta\theta} &= \rho^2 \\
G_{\phi_2\phi_2} &= G\left(1+\hat{\gamma}_2^2\rho_1^2\rho_3^2\right)\rho_2^2\,u^2 & G_{\phi_3\phi_3} &= G\left(1+\hat{\gamma}_3^2\rho_1^2\rho_2^2\right)\rho_3^2\,u^2 \\
G_{\phi_2\phi_3} &= G\,\hat{\gamma}_2\hat{\gamma}_3\,\rho_1^2\rho_2^2\rho_3^2\,u^2 & & \\
G_{\phi_2 X_5} &= -G\,\hat{\gamma}_1\hat{\gamma}_2\,\rho_2^2\rho_3^2\,X_6 & G_{\phi_2 X_6} &= G\,\hat{\gamma}_1\hat{\gamma}_2\,\rho_2^2\rho_3^2\,X_5 \\
G_{\phi_3 X_5} &= -G\,\hat{\gamma}_1\hat{\gamma}_3\,\rho_2^2\rho_3^2\,X_6 & G_{\phi_3 X_6} &= G\,\hat{\gamma}_1\hat{\gamma}_3\,\rho_2^2\rho_3^2\,X_5 \\
G_{X_5X_5} &= 1-\frac{X_6^2}{u^2\rho_1^2}\left[1-G\left(1+\hat{\gamma}_1^2\rho_2^2\rho_3^2\right)\right] & G_{X_6X_6} &= 1-\frac{X_5^2}{u^2\rho_1^2}\left[1-G\left(1+\hat{\gamma}_1^2\rho_2^2\rho_3^2\right)\right] \\
G_{X_5X_6} &= \frac{X_5X_6}{u^2\rho_1^2}\left[1-G\left(1+\hat{\gamma}_1^2\rho_2^2\rho_3^2\right)\right] & &
\end{aligned}
\tag{2.5}
$$

where G is given in (2.1) and now

$$
\rho_1^2=\frac{X_5^2+X_6^2}{u^2}\quad,\quad \rho_2^2=\frac{\rho^2c_\theta^2}{u^2}\quad,\quad \rho_3^2=\frac{\rho^2s_\theta^2}{u^2} \tag{2.6}
$$

The constraint $\sum_{i=1}^3\rho_i^2=1$ is traded with the condition $u^2=\rho^2+X_5^2+X_6^2$.

The LM-Frolov supergravity solution is characterized by a non-constant dilaton

$$
e^{2\phi}=e^{2\phi_0}G \tag{2.7}
$$

where ϕ_0 is the constant dilaton of the undeformed background related to the AdS radius by $R^4=4\pi e^{\phi_0}N\equiv\lambda$. For real deformation parameters $\hat{\gamma}_i$ the axion field C_0 is a constant and can be set to zero.

This background carries also a non-vanishing NS-NS two-form and R-R forms as well. In our set of coordinates they read

$$
\begin{aligned}
B &= \frac{R^2G}{u^2}\left((X_5dX_6-X_6dX_5)\wedge(\hat{\gamma}_3\rho_2^2d\phi_2-\hat{\gamma}_2\rho_3^2d\phi_3)+\hat{\gamma}_1\rho_2^2\rho_3^2\,u^2d\phi_2\wedge d\phi_3\right) \\
C_2 &= 4R^2e^{-\phi_0}\omega_1\wedge\left(\hat{\gamma}_1\frac{X_5dX_6-X_6dX_5}{u^2\rho_1^2}+\hat{\gamma}_2d\phi_2+\hat{\gamma}_3d\phi_3\right)\quad,\qquad \omega_1=\frac{\rho^4}{4u^4}c_\theta s_\theta d\theta \\
C_4 &= 4R^4e^{-\phi_0}\left(\frac{u^4}{4R^8}dt\wedge dx_1\wedge dx_2\wedge dx_3\ -G\,\omega_1\wedge\frac{X_5dX_6-X_6dX_5}{u^2\rho_1^2}\wedge d\phi_2\wedge d\phi_3\right)
\end{aligned}
\tag{2.8}
$$

The corresponding field strengths are given by the general prescription $\tilde{F}_q=dC_{q-1}-dB\wedge C_{q-3}$.

The missing forms of higher degrees can be found by applying the ten-dimensional Hodge duality operator

$$
\tilde{F}_7=-\star\tilde{F}_3,\qquad\qquad \tilde{F}_9=\star\tilde{F}_1 \tag{2.9}
$$

From the first identity and using the equation of motion for C_2

$$
d(\star\tilde{F}_3)=dC_4\wedge dB, \tag{2.10}
$$

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it is easy to see that $d(C_6 - B \wedge C_4) = 0$, i.e. $C_6 - B \wedge C_4 = dX$ for an arbitrary 5-form X. We make the gauge choice

$$C_6 = C_4 \wedge B \tag{2.11}$$

Finally, from the second identity in (2.9), by using (2.11) and taking into account that $B \wedge B = 0$ and $C_0 = 0$ we find $\tilde{F}_9 = dC_8 = 0$. Therefore, in what follows we set $C_8 = 0$.

The deformed background written in terms of the original internal coordinates $(\rho, \alpha, \theta, \phi_1, \phi_2, \phi_3)$ has a manifest invariance under constant shifts of the toroidal coordinates (ϕ_1, ϕ_2, ϕ_3) which correspond to three $\mathrm{U}(1)$ symmetries. With our choice of coordinates the invariance under $\phi_{2,3} \to \phi_{2,3} + \text{const.}$ is still manifest, whereas the third $\mathrm{U}(1)$ associated to shifts of ϕ_1 is realized as a rotation in the (X_5, X_6) plane.

3. The embedding of D7-branes

We now study the embedding of $N_f \ll N$ D7-branes in the deformed background described in the previous section. For simplicity we consider the case of a single spacetime filling D7-brane ($N_f = 1$) which extends in the internal directions $(\rho, \theta, \phi_2, \phi_3)$ (we work in the static gauge where the worldvolume coordinates σ^a of the brane are identified with the appropriate ten dimensional coordinates). The X_5, X_6 coordinates parametrize the mutual orthogonal directions of the intersecting system of N sources D3-branes and one flavor D7-brane.

The dynamics of bosonic degrees of freedom of the D7-brane is described by the action

$$S = S_{DBI} + S_{\mathrm{WZ}} \tag{3.1}$$

where S_{DBI} is the abelian Dirac-Born-Infeld term (in what follows latin labels $a, b, \ldots$ stand for worldvolume components)

$$S_{DBI} = -T_7 \int_{\Sigma_8} d^8\sigma \, e^{-\phi} \sqrt{-det(g_{ab} + \mathcal{F}_{ab})} \tag{3.2}$$

whereas S_{WZ} is the Wess-Zumino term describing the coupling of the brane to the R-R potentials

$$S_{\mathrm{WZ}} = T_7 \int_{\Sigma_8} \left\{ \frac{(2\pi\alpha')^3}{6} P[C_2] \wedge F \wedge F \wedge F + \frac{(2\pi\alpha')^2}{2} P[C_4 - C_2 \wedge B] \wedge F \wedge F \right\} \tag{3.3}$$

Here $g_{ab} \equiv G_{MN} \partial_a X^M \partial_b X^N$ is the pull-back of the ten-dimensional spacetime metric (2.4), (2.5) on the worldvolume Σ_8 and T_7 is the D7-brane tension. The $\mathrm{U}(1)$ worldvolume gauge field strength F_{ab} enters the action through the modified field strength $\mathcal{F}_{ab} = 2\pi\alpha' F_{ab} - b_{ab}$, where b_{ab} is the pull-back of the target NS-NS two-form potential in (2.8), $b_{ab} = B_{MN} \partial_a X^M \partial_b X^N$. Moreover, in (3.3) $P[\ldots]$ denotes the pull-back of the R-R forms on Σ_8.

We look for ground state configurations of the D7-brane. These are static solutions of the equations of motion for X_5, X_6 and εF ($\varepsilon \equiv 2\pi\alpha'$) derived from (3.1).

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In the ordinary $\mathrm{AdS}_5 \times \mathrm{S}^5$ background static embeddings (see for example [13]) can be found by setting $X_6 = 0$, $F = 0$ and $X_5 = X_5(\rho)$ satisfying

$$\frac{d}{d\rho}\left(\frac{\rho^3}{\sqrt{1+(\partial_\rho X_5)^2}}\frac{dX_5}{d\rho}\right) = 0 \tag{3.4}$$

with asymptotic behavior $X_5(\rho) = L + \frac{c}{\rho^2}$ for $\rho \gg 1$. The mass solution $X_5 = L$ is the only well-behaved solution and corresponds to fixing the location of the D7-brane in the 56-plane at $X_5^2 + X_6^2 = L^2$. This is a BPS configuration since the energy density turns out to be independent of L [30, 12].

In the deformed background we consider an embedding of the form

$$X^M = (x_\mu, \rho, \theta, \phi_2, \phi_3, X_5(\rho), X_6(\rho)) \quad , \quad F = F(X^M) \tag{3.5}$$

where, as in the ordinary case, we allow for a non-trivial dependence of the orthogonal directions on the non-compact internal coordinate ρ. Solving the equations of motion for X_5, X_6 and F in the present case requires a bit of care since the non-vanishing NS-NS 2-form in (2.8) can act as a source for the field strength εF.

We expand the action (3.1) up to second order in εF. The WZ action is simply

$$S_{\mathrm{WZ}} = \frac{T_7}{2}\int_{\Sigma_8} P[C_4 - C_2 \wedge B] \wedge \varepsilon F \wedge \varepsilon F \tag{3.6}$$

whereas the expansion of S_{DBI} gives

$$\begin{aligned}
\mathcal{L}_{DBI} &= -T_7 \frac{\sqrt{-det(g-b+\varepsilon F)}}{\sqrt{G}} \\
&= -T_7 \frac{\sqrt{-det(g-b)}}{\sqrt{G}}\sqrt{det(1+Y)} \\
&= -T_7\, \rho^3 s_\theta c_\theta \sqrt{\Omega_2}\left\{1 + \frac{1}{2}\mathrm{Tr}(Y) - \frac{1}{4}\mathrm{Tr}(Y^2) + \frac{1}{8}\left[\mathrm{Tr}(Y)\right]^2 + \cdots\right\}
\end{aligned} \tag{3.7}$$

where we have defined

$$\begin{aligned}
Y &\equiv (g-b)^{-1}\,\varepsilon F \\
\Omega_2 &\equiv 1 + (\partial_\rho X_5)^2 + (\partial_\rho X_6)^2
\end{aligned} \tag{3.8}$$

and set $e^{\phi_0} \equiv 1$.

The source for εF comes from the term

$$\frac{1}{2}\mathrm{Tr}(Y) = \frac{\varepsilon}{R^2\Omega_2}\left[(X_5\,\partial_\rho X_6 - X_6\,\partial_\rho X_5)(\hat{\gamma}_2\, F_{\rho\phi_3} - \hat{\gamma}_3\, F_{\rho\phi_2}) - \hat{\gamma}_1\Omega_2\, F_{\phi_2\phi_3}\right] \tag{3.9}$$

In the abelian case the last term is a total derivative and, once integrated on the world-volume of the brane, it cancels. We are left with the first term which gives a non-trivial coupling between the scalars and the vectors. We note that these couplings are proportional to the deformation parameters and disappear for $\hat{\gamma}_i = 0$, consistently with the undeformed case.

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Since all the F components except $F_{\rho\phi_2}$ and $F_{\rho\phi_3}$ satisfy homogeneous equations we can set them to zero and concentrate on the system of coupled equations of motion for $X_5, X_6, F_{\rho\phi_2}$ and $F_{\rho\phi_3}$. It is easy to realize that a solution is still given by $X_6 = 0$, $F_{\rho\phi_2} = F_{\rho\phi_3} = 0$, whereas $X_5(\rho)$ satisfies eq. (3.4) and can be chosen as $X_5 = L$.

Therefore, even in the deformed case, the ground state of the probe brane is given by a static location at $X_5^2+X_6^2 = L^2$ with no F flux and absence of non-trivial quark condensate. The choice $X_5 = L$ and $X_6 = 0$ breaks the rotational invariance in the (X_5, X_6) plane.

This configuration is stable (BPS). In fact, the corresponding action

$$S = -T_7 \int_{\Sigma_8} d^8\sigma \rho^3 s_\theta c_\theta \tag{3.10}$$

coincides with the one of the undeformed case and satisfies the no-force condition [30, 12].

Setting $X_5^2 + X_6^2 = L^2$, the induced metric on the D7-brane reads

$$\begin{aligned} ds_I^2 &\equiv g_{ab}\, dX^a dX^b \\ &= \frac{L^2+\rho^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{L^2+\rho^2}(d\rho^2 + \rho^2 d\theta^2) \\ &\quad + \frac{R^2 G \rho^2}{(L^2+\rho^2)}\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\rho^2 L^2 c_\theta^2 s_\theta^2 (\hat\gamma_2 d\phi_2 + \hat\gamma_3 d\phi_3)^2}{(L^2+\rho^2)^2}\right] \end{aligned} \tag{3.11}$$

where G in (2.1) takes the explicit form

$$G = \frac{(L^2+\rho^2)^2}{(L^2+\rho^2)^2 + \hat\gamma_1^2 \rho^4 s_\theta^2 c_\theta^2 + \hat\gamma_2^2 L^2 \rho^2 s_\theta^2 + \hat\gamma_3^2 L^2 \rho^2 c_\theta^2} \tag{3.12}$$

We note that, due to the particular embedding we have realized, the parameter $\hat\gamma_1$ associated to the TsT transformation on the (ϕ_2, ϕ_3) torus inside the D7 worldvolume enters the metric differently from $\hat\gamma_{2,3}$ which are instead associated to deformations on tori with one parallel and one orthogonal direction to the probe.

The different role played by $\hat\gamma_1$ respect to $(\hat\gamma_2, \hat\gamma_3)$ can be also understood by looking at the conformal case ($L = 0$) or the UV limit ($\rho \to \infty$) of the theory. In both cases the dependence on $(\hat\gamma_2, \hat\gamma_3)$ disappears and the worldvolume metric reduces to the one for $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^3$ where $\tilde{\mathrm{S}}^3$ is the deformed three-sphere with metric

$$\frac{ds^2_{\tilde{\mathrm{S}}^3}}{R^2} = d\theta^2 + G(c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2) \quad , \quad G = \frac{1}{1+\hat\gamma_1^2 c_\theta^2 s_\theta^2} \tag{3.13}$$

Instead, for ρ finite and $L \neq 0$ the AdS_5 factor is lost, the theory is no longer conformal and a non-trivial dependence on all the deformation parameters appears.

The particular probe brane configuration we have chosen is smoothly related to the one of the undeformed case. In fact, sending $\hat\gamma_i \to 0$ we recover the usual Karch-Katz [2] picture of flavor branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$. As we have just proved, the stability of the D3-D7 system survives the deformation.

We have embedded flavor D7-branes in a deformed background. When the D7-brane is spacetime filling and wraps the (ϕ_2, ϕ_3) torus the configuration is stable and no worldvolume flux is turned on. Alternatively, we could have started with a configuration of

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D7-branes in the undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and perform the three TsT transformations as a second step. If the D7-branes were to be placed along the same directions as before, we would obtain exactly the same configuration of stable D7-branes in the deformed background with no flux turned on. In fact, along the directions (ϕ_1, ϕ_2, ϕ_3) affected by TsT transformations the probe branes have Dirichlet-Neumann-Neumann (DNN) boundary conditions. Considering the proposal in [25] and according to the analysis of [27] a DNN configuration with no flux is mapped into the same configuration, whatever is the TsT transformation we perform. Therefore, for the particular embedding we are analyzing the two operations i) Adding a probe to the deformed background and ii) Performing a TsT transformation on the undeformed brane scenario are equivalent processes. The stability of our brane configuration for any value of the deformation parameters then follows from the fact that TsT transformations do not affect the BPS nature of the original brane system [21] (see also [26]).

It is worth stressing that the possibility of applying equivalently prescriptions i) or ii) is peculiar of the particular brane configuration we have chosen. Had we considered different embeddings, the two procedures wouldn't had led necessarily to equivalent settings [25, 27]. Furthermore, the stability of the configuration would have become questionable.

When the deformation parameters $\hat{\gamma}_i$ are all equal the $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^5$ background has $\mathcal{N} = 1$ supersymmetry. The question is whether our D7-brane embedding preserves supersymmetry. The standard way of finding supersymmetric configurations is to look at the κ-symmetry condition of the probes. However, since the β-deformed background can be described by an SU(2) structure manifold, it is more convenient to work using the formalism of G-structures [31] and Generalized Complex Geometry (GCG) [32]. In this framework the supersymmetry conditions for D-branes probing SU(2) structure manifolds have been established in [28]. For spacetime filling D7-branes a class of supersymmetric embeddings is given by $z_1 \equiv X_5 + iX_6 = L$, with $z_2 \equiv X_1 + iX_2$ and $z_3 \equiv X_3 + iX_4$ arbitrarily fixed and no worldvolume flux turned on. This embeddings break one of the $\mathrm{U}(1)$ global symmetries. Since our configuration belongs to this class we conclude that our embedding is supersymmetric.

4. Probe fluctuations

As proposed in [3, 4] D7-brane fluctuations around its ground state are dual to color singlets which may be interpreted as describing mesonic states of the four dimensional gauge theory. The mass spectrum of the mesons is given by the Kaluza-Klein spectrum of states which originate from the compactification of the D7-brane on the internal submanifold. In the ordinary undeformed scenario the spectrum is discrete and with a mass gap [5].

Our main purpose is to investigate probe fluctuations in the deformed background.

A generic vibration of the brane around its ground state can be described by

$$X_5 = L + \varepsilon\, \chi(\sigma^a)\,, \qquad\qquad X_6 = \varepsilon\, \varphi(\sigma^a) \tag{4.1}$$

together with a non-trivial flux $\varepsilon F_{ab} = \varepsilon(\partial_a A_b - \partial_b A_a)$. The fluctuations are functions of the worldvolume coordinates σ^a and ε is a small perturbation parameter.

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We expand the action of the probe brane in powers of the small parameter

$$S = S_{DBI} + S_{\rm WZ} = \int_{\Sigma_8} d^8\sigma \{\mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots\} \tag{4.2}$$

and consider terms up to the quadratic order in ε.

We first concentrate on the DBI term

$$\mathcal{L}_{DBI} = -T_7 \frac{1}{\sqrt{G}} \sqrt{-det(g - b + \varepsilon F)} \tag{4.3}$$

where we have written the dilaton field as in (2.7) with $e^{\phi_0} \equiv 1$.

We expand the various terms by writing

$$\begin{aligned} g &= g^{(0)} + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)}\,, \qquad b = b^{(0)} + \varepsilon b^{(1)} + \varepsilon^2 b^{(2)} \\ \frac{1}{\sqrt{G}} &= G^{(0)} + \varepsilon G^{(1)} + \varepsilon^2 G^{(2)} \end{aligned} \tag{4.4}$$

Therefore, the determinant can be written as

$$\begin{aligned} \sqrt{-det(g-b+\varepsilon F)} &= \sqrt{-det\left(g^{(0)} - b^{(0)}\right)}\sqrt{det(1+Y)} \\ &= \sqrt{-det\left(g^{(0)} - b^{(0)}\right)}\left[1+\frac{1}{2}\mathrm{Tr}(Y)-\frac{1}{4}\mathrm{Tr}(Y^2)+\frac{1}{8}\left[\mathrm{Tr}(Y)\right]^2+\cdots\right] \end{aligned} \tag{4.5}$$

where the matrix Y is given by

$$Y = \left(g^{(0)} - b^{(0)}\right)^{-1}\left[\varepsilon\left(g^{(1)} - b^{(1)} + F\right) + \varepsilon^2\left(g^{(2)} - b^{(2)}\right) + \cdots\right] \tag{4.6}$$

At the lowest order the contribution $g^{(0)}$ is easily read from (3.11), whereas for the pull-back of B from eq. (2.8) we find that the only non-vanishing component is $b^{(0)}_{\phi_2\phi_3} = \hat{\gamma}_1 R^2 G \rho_2^2 \rho_3^2$.

It is convenient to introduce the undeformed induced metric

$$\mathcal{G} = diag\left(-\frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{R^2}{L^2+\rho^2}, \frac{R^2\rho^2}{L^2+\rho^2}, \frac{R^2\rho^2 c_\theta^2}{L^2+\rho^2}, \frac{R^2\rho^2 s_\theta^2}{L^2+\rho^2}\right) \tag{4.7}$$

the auxiliary metric $\mathcal{C}$ defined by

$$\begin{aligned} d\hat{s}^2 &\equiv \mathcal{C}_{ab} d\sigma^a d\sigma^b \\ &= \frac{L^2+\rho^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{L^2+\rho^2}(d\rho^2 + \rho^2 d\theta^2) \\ &\quad + \frac{R^2\hat{G}\rho^2}{L^2+\rho^2}\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\rho^2 L^2 c_\theta^2 s_\theta^2 (\hat{\gamma}_2 d\phi_2 + \hat{\gamma}_3 d\phi_3)^2}{(L^2+\rho^2)^2}\right] \end{aligned} \tag{4.8}$$

with

$$\hat{G} = \frac{(L^2+\rho^2)^2}{(L^2+\rho^2)^2 + \hat{\gamma}_2^2 L^2\rho^2 s_\theta^2 + \hat{\gamma}_3^2 L^2 \rho^2 c_\theta^2} \tag{4.9}$$

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and two deformation matrices $\mathcal{T}$ and $\mathcal{J}$ given by

$$\begin{aligned} \mathcal{T}^{\phi_2\phi_2} &= \hat{\gamma}_3^2 \qquad \mathcal{T}^{\phi_3\phi_3} = \hat{\gamma}_2^2 \qquad \mathcal{T}^{\phi_2\phi_3} = \mathcal{T}^{\phi_3\phi_2} = -\hat{\gamma}_2\hat{\gamma}_3 \\ \mathcal{J}^{\phi_2\phi_2} &= 0 \qquad \mathcal{J}^{\phi_3\phi_3} = 0 \qquad \mathcal{J}^{\phi_2\phi_3} = -\mathcal{J}^{\phi_3\phi_2} = \gamma_1 \end{aligned} \tag{4.10}$$

The metric $\mathcal{C}$ is nothing but the induced metric (3.11) evaluated at $\hat{\gamma}_1 = 0$. Its inverse can be expressed as

$$\mathcal{C}^{-1} = \mathcal{G}^{-1} + \frac{L^2}{R^2(L^2+\rho^2)}\mathcal{T} \tag{4.11}$$

It turns out that the matrix $\left(g^{(0)} - b^{(0)}\right)^{-1}$ in (4.6) can be written as

$$\left(g^{(0)} - b^{(0)}\right)^{-1} = \mathcal{C}^{-1} + \mathcal{J} = \mathcal{G}^{-1} + \frac{L^2}{R^2(L^2+\rho^2)}\mathcal{T} + \mathcal{J} \tag{4.12}$$

Since the whole dependence on the deformation parameters is encoded in $\mathcal{T}$ and $\mathcal{J}$, the $\hat{\gamma}_i \to 0$ limit is easily understood.

Now a long but straightforward calculation allows to determine the first order corrections $g^{(1)}, b^{(1)}, G^{(1)}$ as well as the second order ones $g^{(2)}, b^{(2)}, G^{(2)}$. Inserting in $\mathcal{L}_{DBI}$ we eventually find

$$\begin{aligned} \mathcal{L}_{DBI}^{(0)} &= -T_7\rho^3 c_\theta s_\theta \\ \mathcal{L}_{DBI}^{(1)} &= T_7\rho^3 c_\theta s_\theta \hat{\gamma}_1 F_{\phi_2\phi_3}/R^2 \\ \mathcal{L}_{DBI}^{(2)} &= -T_7\rho^3 c_\theta s_\theta \left[\frac{R^2}{2(L^2+\rho^2)}\mathcal{C}^{ab}\partial_a\chi\partial_b\chi + \frac{R^2}{2(L^2+\rho^2)}\mathcal{G}^{ab}\partial_a\varphi\partial_b\varphi \right. \\ &\qquad \left. + \frac{1}{4}F_{ab}F^{ab} + \frac{L}{(L^2+\rho^2)}(\hat{\gamma}_2 F_{a\phi_3} - \hat{\gamma}_3 F_{a\phi_2})\,\mathcal{G}^{ab}\partial_b\varphi\right] \end{aligned} \tag{4.13}$$

where $F^{ab} \equiv \mathcal{C}^{ac}\mathcal{C}^{bd}F_{cd}$ and $\mathcal{C}^{ac}$ is given in (4.11). The first order Lagrangian is a total derivative since our embedding $X_5 = L$, $X_6 = 0$ is an exact solution of the equations of motion.

The Wess-Zumino Lagrangian starts with a second order term in ε given by

$$\mathcal{L}_{\rm WZ} = T_7\,\frac{1}{2}P[C_4 - C_2\wedge B]\wedge F\wedge F = T_7\frac{(L^2+\rho^2)^2}{R^4}\,\epsilon^{ijk}\partial_\rho A_i\partial_j A_k \tag{4.14}$$

where we use latin indices to indicate coordinates on the three-sphere parametrized by (θ, ϕ_2, ϕ_3), A_i is the flux potential on it and ϵ^{ijk} is the Levi-Civita tensor density ($\epsilon^{\theta 23} = 1$). This term turns out to be independent of the deformation parameters since the combination $(C_4 - C_2\wedge B)$ at lowest order gives exactly the 4-form of the $\mathrm{AdS}_5\times\mathrm{S}^5$ undeformed geometry.

Determining the equations of motion from the previous Lagrangian is now an easy task. Introducing the fixed vector

$$v^a = \hat{\gamma}_2\delta_3^a - \hat{\gamma}_3\delta_2^a \tag{4.15}$$

for the χ and φ scalars we find

$$\partial_a\left[\sqrt{-det(\mathcal{G})}\,\left(\frac{R^2}{(L^2+\rho^2)}\,\mathcal{G}^{ab} + \frac{L^2}{(L^2+\rho^2)^2}v^a v^b\right)\partial_b\chi\right] = 0 \tag{4.16}$$

$$\partial_a\left[\sqrt{-det(\mathcal{G})}\,\frac{R^2}{(L^2+\rho^2)}\,\mathcal{G}^{ab}\left(\partial_b\varphi + \frac{L}{R^2}v^c F_{bc}\right)\right] = 0 \tag{4.17}$$

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whereas, using (4.17) the equations of motion for the gauge fields take the form

$$\partial_a \left[\sqrt{-det(\mathcal{G})}\, \mathcal{G}^{ac}\mathcal{G}^{bd}\, F_{cd}\right] - \frac{4\rho(L^2+\rho^2)}{R^4}\epsilon^{bjk}\partial_j A_k \\ - \sqrt{-det(\mathcal{G})}\, \frac{L}{(L^2+\rho^2)}\, v^d \partial_d \left[\mathcal{G}^{bc}\left(\partial_c\varphi + \frac{L}{R^2} v^f F_{cf}\right)\right] = 0 \tag{4.18}$$

It is interesting to note that the equations of motion depend only on the deformation parameters $\hat{\gamma}_2$ and $\hat{\gamma}_3$ hidden in the vector v. In fact, at this order the dependence on the parameter $\hat{\gamma}_1$ associated to the torus inside the D7 worldvolume completely cancels between the factors $\sqrt{-det(g-b+\varepsilon F)}$ and $1/\sqrt{G}$.

The scalar fluctuation χ along the direction where the branes are located at distance L decouples from the rest. The scalar φ, instead, interacts non-trivially with the worldvolume gauge fields through terms proportional to the deformation parameters.

The vector v has non-vanishing components only on the three-sphere and selects there a fixed direction. As a consequence, the equations of motion (4.16)–(4.18) loose SO(4) invariance.

As a first application we consider the $L = 0$ conformal case. The vibration of the brane is given by $X_5 = \varepsilon\,\chi(\sigma^a)$ and $X_6 = \varepsilon\,\varphi(\sigma^a)$. The equations of motion reduce to

$$\partial_a \left[\sqrt{-det(\mathcal{G})}\, \frac{R^2}{\rho^2}\, \mathcal{G}^{ab}\, \partial_b \Psi\right] = 0$$
$$\partial_a \left[\sqrt{-det(\mathcal{G})}\, \mathcal{G}^{ac}\mathcal{G}^{bd}\, F_{cd}\right] - \frac{4\rho^3}{R^4}\epsilon^{bjk}\partial_j A_k = 0\,. \tag{4.19}$$

where $\Psi \equiv (\varphi, \chi)$ and $\mathcal{G}^{ab}$ is the inverse of the matrix (4.7) evaluated at $L = 0$. We see that the dependence on the deformation parameters disappears completely and the equations of motion reduce to the ones of the undeformed case [5]. In particular, the scalar and gauge fluctuations decouple. Written explicitly, the scalar equations read

$$\frac{R^4}{\rho^4}\partial^\mu\partial_\mu\Psi + \frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho\Psi) + \frac{1}{\rho^2}\Delta_{\mathrm{S}^3}\Psi = 0 \tag{4.20}$$

where

$$\Delta_{\mathrm{S}^3}\Psi \equiv \frac{1}{c_\theta s_\theta}\partial_\theta(c_\theta s_\theta \partial_\theta\Psi) + \frac{1}{c_\theta^2}\partial_2^2\Psi + \frac{1}{s_\theta^2}\partial_3^2\Psi \tag{4.21}$$

is the Laplacian on the unit 3-sphere ($\partial_2 \equiv \partial_{\phi_2}$, $\partial_3 \equiv \partial_{\phi_3}$).

According to the results in [2, 5] the corresponding AdS_5 masses are above the Breitenlohner-Freedman bound [33]. This is a further check of the stability of our brane configuration.

5. The mesonic spectrum

We now concentrate on the more general situation $X_5 = L + \varepsilon\,\chi(\sigma^a)$, $X_6 = \varepsilon\,\varphi(\sigma^a)$ and solve the equations of motion (4.16)–(4.18) for scalar and vector modes. We write the abelian flux in terms of its potential one-form, $F_{ab} = \partial_a A_b - \partial_b A_a$, and choose the Lorentz gauge $\partial_\mu A^\mu = 0$ on the spacetime components.

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We find convenient to introduce covariant derivatives on the unit three-sphere (θ, ϕ_2, ϕ_3). Given its metric $g = diag(1, c_\theta^2, s_\theta^2)$, we have $\nabla_i V^j = \partial_i V^j + \Gamma^j_{ik} V^k$ with the only non-vanishing components being $\Gamma^\theta_{22} = -\Gamma^\theta_{33} = c_\theta s_\theta$, $\Gamma^2_{2\theta} = -\frac{s_\theta}{c_\theta}$ and $\Gamma^3_{3\theta} = \frac{c_\theta}{s_\theta}$.

In order to simplify the equations we introduce the special operators

$$\begin{aligned}
\mathcal{O}_{\hat\gamma} &\equiv \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu + \frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho) + \frac{1}{\rho^2}\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i) + \frac{L^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)^2 \\
\tilde{\mathcal{O}}_{\hat\gamma} &\equiv \frac{R^4}{(L^2+\rho^2)^2}\,\partial^\nu\partial_\nu + \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_\rho\right] + \frac{1}{\rho^2}\nabla_l\nabla^l \\
&\quad + \frac{L^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)^2
\end{aligned} \tag{5.1}$$

along with their undeformed versions $\mathcal{O}_0 \equiv \mathcal{O}_{\hat\gamma}|_{\hat\gamma_2=\hat\gamma_3=0}$, $\tilde{\mathcal{O}}_0 \equiv \tilde{\mathcal{O}}_{\hat\gamma}|_{\hat\gamma_2=\hat\gamma_3=0}$.

Equation (4.16) for the χ mode then takes the compact form

$$\mathcal{O}_{\hat\gamma}\,\chi = 0 \tag{5.2}$$

whereas equation (4.17) can be rewritten as

$$\mathcal{O}_0\,\Phi - \frac{L}{R^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l\right] = 0 \tag{5.3}$$

where we have defined

$$\Phi \equiv \varphi + \frac{L}{R^2}v^a A_a = \varphi + \frac{L}{R^2}(\hat\gamma_2 A_3 - \hat\gamma_3 A_2) \tag{5.4}$$

Equations (4.18) for the vector modes come into three classes, according to b being in Minkowski, or $b = \rho$ or $b = i \equiv \{\theta,\, \phi_2,\, \phi_3\}$. We list the three cases.

- b in Minkowski: For $b = \mu$ and expressing the F flux in terms of its one-form potential, equation (4.18) becomes

$$\mathcal{O}_{\hat\gamma}\,A_\mu - \partial_\mu\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\,\Phi\right] = 0 \tag{5.5}$$

  with Φ defined in (5.4).

  We apply ∂^μ to this equation and sum over μ. Using $[\partial^\mu, \mathcal{O}_{\hat\gamma}] = 0$ and Lorentz gauge, solutions corresponding to non-trivial dispersion relations ($k^2 \neq 0$) satisfy

$$\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\Phi\right] = 0 \quad , \quad \mathcal{O}_{\hat\gamma}\,A_\mu = 0 \tag{5.6}$$

- $b = \rho$: Again, expressing the flux in terms of the vector potential we obtain

$$\mathcal{O}_{\hat\gamma}\,A_\rho - \left[\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\rho) + \frac{1}{\rho^2}\partial_\rho\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\partial_\rho\Phi\right] = 0 \tag{5.7}$$

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- $b = i$: On the internal $\tilde{\mathrm{S}}^3$ sphere we have

$$\tilde{\mathcal{O}}_{\hat{\gamma}}\, A_j - \frac{1}{\rho^2}\left(\nabla_l \nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) \\ - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_j A_\rho\right] - \frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2\partial_3 - \hat{\gamma}_3\partial_2)\,\partial_j\Phi = 0 \tag{5.8}$$

where we have used $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^{ij}) = \nabla_i F^{ij} = \nabla_i\nabla^i A^j - \nabla_i\nabla^j A^i$.

Now, collecting all the equations and using the first of (5.6) in (5.3) the system of coupled equations we need solve is

$$\begin{aligned}
&(0) \quad \mathcal{O}_{\hat{\gamma}}\chi = 0\,; \qquad\qquad \mathcal{O}_{\hat{\gamma}}\, A_\mu = 0 \\
&(1) \quad \mathcal{O}_{\hat{\gamma}}\,\Phi = 0 \\
&(2) \quad \left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla^l A_l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2\partial_3 - \hat{\gamma}_3\partial_2)\,\Phi\right] = 0 \\
&(3) \quad \mathcal{O}_{\hat{\gamma}}\, A_\rho - \left[\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\rho) + \frac{1}{\rho^2}\partial_\rho\nabla^l A_l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2\partial_3 - \hat{\gamma}_3\partial_2)\partial_\rho\Phi\right] = 0 \\
&(4) \quad \tilde{\mathcal{O}}_{\hat{\gamma}}\, A_j - \frac{1}{\rho^2}\left(\nabla_l\nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) \\
&\qquad - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_j A_\rho\right] - \frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2\partial_3 - \hat{\gamma}_3\partial_2)\partial_j\Phi = 0
\end{aligned} \tag{5.9}$$

Equations (1)–(4) exhibit a non-trivial interaction between the scalar Φ and the components of the vector potential along the internal directions. The modes χ and A_μ instead decouple.

It is convenient to search for solutions expanded in spherical harmonics on S^3. Scalar spherical harmonics are a complete set of functions $\mathcal{Y}_l^{m_2,m_3}$ in the $\left(\frac{l}{2},\frac{l}{2}\right)$ representation of SO(4) and with definite $\mathrm{U}(1)\times\mathrm{U}(1)$ quantum numbers (m_2, m_3) satisfying $|m_2+m_3| = |m_2-m_3| = l-2k$, $l,k = 0,1,\ldots$. For fixed l the degeneracy is $(l+1)^2$. Their defining equations are[4]

$$\begin{aligned}
\Delta_{\mathrm{S}^3}\,\mathcal{Y}_l^{m_2,m_3} &= -l(l+2)\,\mathcal{Y}_l^{m_2,m_3} \\
\frac{\partial}{\partial\phi_{2,3}}\,\mathcal{Y}_l^{m_2,m_3} &= i m_{2,3}\,\mathcal{Y}_l^{m_2,m_3}
\end{aligned} \tag{5.10}$$

Vector spherical harmonics come into three classes. Choosing them to be also eigenfunctions of $\frac{\partial}{\partial\phi_{2,3}}$ we have longitudinal harmonics $\mathcal{H}_i = \nabla_i\mathcal{Y}_l^{m_2,m_3}$, $l\geq 1$ which are in the $(\frac{l}{2},\frac{l}{2})$ representation of SO(4) with (m_2,m_3) ranging as before. Transverse harmonics are $\mathcal{M}_i^+ \equiv \mathcal{Y}_i^{(l,m_2,m_3);+}$ with $l\geq 1$ in the $\left(\frac{l-1}{2},\frac{l+1}{2}\right)$ and $\mathcal{M}_i^- \equiv \mathcal{Y}_i^{(l,m_2,m_3);-}$ with $l\geq 1$ in the $\left(\frac{l+1}{2},\frac{l-1}{2}\right)$. Their degeneracy is $l(l+2)$ and it is counted by $|m_2+m_3| = l\pm 1-2k$, $|m_2-m_3| = l\mp 1-2k$.

[4]For their explicit realization see for instance [34, 24].

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These harmonics satisfy

$$\begin{aligned}
\nabla_i \nabla^i \mathcal{M}^{\pm}_j - R^k_j \mathcal{M}^{\pm}_k &= -(l+1)^2\, \mathcal{M}^{\pm}_j \\
\epsilon_{ijk} \nabla^j \mathcal{M}^{\pm;k} &= \pm \sqrt{g}\,(l+1)\, \mathcal{M}^{\pm}_i \\
\nabla^i \mathcal{M}^{\pm}_i &= 0 \\
\frac{\partial}{\partial \phi_{2,3}}\, \mathcal{M}^{\pm}_i &= i m_{2,3}\, \mathcal{M}^{\pm}_i
\end{aligned} \tag{5.11}$$

where $\sqrt{g} = c_\theta s_\theta$ is the square root of the determinant of the metric on S^3, whereas $R^i_j = 2\delta^i_j$ is the Ricci tensor.

As in the undeformed case [5] we require the solutions to be regular at the origin ($\rho = 0$), normalizable and small enough to justify the quadratic approximation. All these conditions are used to select the actual mass spectrum of the mesonic excitations.

5.1 The decoupled modes

5.1.1 The scalar mode χ

We start solving the equation for the decoupled scalar χ. Using the general identity $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i s) = \nabla_i \nabla^i s$ valid for any scalar s, the equation $\mathcal{O}_{\hat{\gamma}}\chi = 0$ reads explicitly

$$\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu \partial_\nu \chi + \frac{1}{\rho^3}\partial_\rho(\rho^3 \partial_\rho \chi) + \frac{1}{\rho^2}\nabla_l \nabla^l \chi + \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 \partial_3 - \hat{\gamma}_3 \partial_2)^2 \chi = 0 \tag{5.12}$$

We look for single-mode solutions of the form

$$\chi(\sigma^a) = r(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta, \phi_2, \phi_3) \tag{5.13}$$

Inserting in (5.12) we obtain an equation for $r(\rho)$ that, after the redefinitions

$$\varrho = \frac{\rho}{L}\,, \qquad \hat{\Gamma}^2 = -\frac{k^2 R^4}{L^2} - (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2 = \bar{M}^2 - (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\,, \tag{5.14}$$

becomes

$$\partial^2_\varrho r + \frac{3}{\varrho}\partial_\varrho r + \left[\frac{\hat{\Gamma}^2}{(1+\varrho^2)^2} - \frac{l(l+2)}{\varrho^2}\right] r = 0 \tag{5.15}$$

This has exactly the same structure of the equation found in the undeformed case [5]. The only difference is the presence of the deformation parameters in $\hat{\Gamma}^2$ which in the undeformed case reduces simply to $\bar{M}^2$. Following what has been done in that case [5] we find that the general solution is

$$r(\rho) = \rho^l (L^2+\rho^2)^{-\alpha} F(-\alpha, -\alpha+l+1; l+2; -\rho^2/L^2) \tag{5.16}$$

where F is the hypergeometric function and $\alpha = \frac{-1+\sqrt{1+\hat{\Gamma}^2}}{2}$. This solution satisfies the conditions of regularity and normalizability if the quantization condition

$$\hat{\Gamma}^2 = 4(n+l+1)(n+l+2) \qquad n \in N\,, \quad n, l \geq 0 \tag{5.17}$$

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is imposed. Using (5.14) and $M^2 = -k^2$, the mass spectrum of scalar mesons then follows

$$M_\chi(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \tag{5.18}$$

with $n, l \geq 0$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$, k a non-negative integer.

We see that the deformation parameters induce a non-trivial dependence of the mass spectrum on the two U(1) quantum numbers (m_2, m_3), so breaking the degeneracy of the undeformed case.

The mass spectrum is smoothly related to the one of the undeformed case for $\hat{\gamma}_i \to 0$.

5.1.2 The type II modes

We look for excitations of the form

$$A_\mu(\sigma^a) = \zeta_\mu \, Z_{II}(\rho) \, e^{ikx} \, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \quad , \quad k\cdot\zeta = 0 \tag{5.19}$$

Following the classification introduced in [5] for the undeformed case we call them Type II modes. The equation $\mathcal{O}_{\hat{\gamma}} A_\mu = 0$ in (5.9) yields to

$$\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A_\mu + \frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\mu) + \frac{1}{\rho^2}\nabla_l\nabla^l A_\mu + \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2\partial_3 - \hat{\gamma}_3\partial_2)^2 A_\mu = 0 \tag{5.20}$$

This is exactly the same equation as the one for the scalar mode χ. Therefore, for each component A_μ we follow the same strategy of subsection 5.1.1 and find the mass spectrum

$$M_{II}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \tag{5.21}$$

with $n, l \geq 0$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$.

Even for this type of vector fluctuations the spectrum is smoothly related to the undeformed one for $\hat{\gamma}_i \to 0$.

5.2 The coupled modes

Having performed the field redefinition (5.4) we solve the coupled equations (1)–(4) by considering elementary fluctuations of Φ, A_ρ and A_i.

5.2.1 The type I modes

Being in a different representation the harmonics $\mathcal{M}_i^\pm$ do not mix with the others. Therefore we can make the ansatz[5]

$$\Phi = 0, \qquad A_\rho = 0, \qquad A_i(\sigma^a) = Z_I^\pm(\rho)\, e^{ikx}\, \mathcal{M}_i^\pm(\theta,\phi_2,\phi_3) \tag{5.22}$$

[5] We note that if we were to follow closely the classification of [5] we would call Type I modes the elementary modes with $\varphi = 0$, i.e. with no fluctuations along the X^6 coordinate. However, given the structure of the equations of motion, in our case we find the definition (5.22) more convenient. In any case, the two definitions coincide for $\hat{\gamma}_i = 0$.

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By using the identity $\nabla_i A^i = 0$ as follows from (5.11), equations (1), (2) and (3) in (5.9) are identically satisfied whereas eq. (4) reads

$$\tilde{\mathcal{O}}_{\hat{\gamma}}\, A_j - \frac{1}{\rho^2}\left(\nabla_l \nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) = 0 \tag{5.23}$$

Considering the explicit expression for the operator $\tilde{\mathcal{O}}_{\hat{\gamma}}$ in (5.1) and using properties (5.11) we find that $Z_I^{\pm}(\rho)$ is a solution of the equation

$$\frac{1}{\varrho}\partial_\varrho\left[\varrho(\varrho^2+1)^2\partial_\varrho Z_I^{\pm}\right] + \left[\hat{\Gamma}^2 - \frac{(\varrho^2+1)^2}{\varrho^2}(l+1)^2 \mp 4(\varrho^2+1)(l+1)\right] Z_I^{\pm} = 0 \tag{5.24}$$

where we have used the definitions (5.14). This is formally the same equation as the one of the undeformed case, except for the different definition of $\hat{\Gamma}^2$. Therefore, following the same steps [5] we find that the solutions are still hypergeometric functions

$$\begin{aligned} Z_I^{+}(\rho) &= \rho^{l+1}(\rho^2+L^2)^{-\alpha-1}F(l+2-\alpha,-1-\alpha;l+2;-\rho^2/L^2) \\ Z_I^{-}(\rho) &= \rho^{l+1}(\rho^2+L^2)^{-\alpha-1}F(l-\alpha,1-\alpha;l+2;-\rho^2/L^2) \end{aligned} \tag{5.25}$$

where $\alpha = \frac{-1+\sqrt{1+\hat{\Gamma}^2}}{2}$. Requiring them to be regular at infinity we obtain the following quantization conditions

$$\begin{aligned} \hat{\Gamma}_+^2 &= 4(n+l+2)(n+l+3) \\ \hat{\Gamma}_-^2 &= 4(n+l)(n+l+1) \qquad\qquad n \geq 0 \end{aligned} \tag{5.26}$$

As a consequence the mass spectrum reads

$$\begin{aligned} M_{I,+} &= \frac{2L}{R^2}\sqrt{(n+l+2)(n+l+3)+\left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \qquad \begin{cases} |m_2+m_3| = l-1-2k \\ |m_2-m_3| = l+1-2k \end{cases} \\ M_{I,-} &= \frac{2L}{R^2}\sqrt{(n+l)(n+l+1)+\left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \qquad \begin{cases} |m_2+m_3| = l+1-2k \\ |m_2-m_3| = l-1-2k \end{cases} \end{aligned} \tag{5.27}$$

where $l \geq 1$ and k is a non-negative integer.

5.2.2 The type III modes

Finally, we consider the following fluctuations

$$\begin{aligned} \Phi(\sigma^a) &= X_{III}(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \\ A_\rho(\sigma^a) &= Y_{III}(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \\ A_i(\sigma^a) &= Z_{III}(\rho)\, e^{ikx}\, \nabla_i \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \equiv \nabla_i A(\sigma^a) \end{aligned} \tag{5.28}$$

with $l \geq 1$. We note that $l = 0$ corresponds to having $A_i = 0$. We will comment on this particular case at the end of this section.

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Inserting in (5.9) and using the identities (5.10) for the scalar harmonics, after a bit of algebra the equations (1)–(4) can be rewritten as

$$
\begin{aligned}
&(1) \quad \left[\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu + \frac{1}{\rho^3}\partial_\rho\left(\rho^3\partial_\rho\right) - \frac{l(l+2)}{\rho^2} - \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\right]\Phi = 0 \\
&(2) \quad \frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) - \frac{l(l+2)}{\rho^2}A + i\frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0 \\
&(3) \quad \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A_\rho + \frac{1}{\rho^2}\partial_\rho\left(\frac{1}{\rho}\partial_\rho(\rho^3 A_\rho)\right) \\
&\qquad - \left[\frac{l(l+2)}{\rho^2} + \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\right] A_\rho \\
&\qquad\qquad + 2iLR^2\frac{(L^2-\rho^2)}{\rho(L^2+\rho^2)^3}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0 \\
&(4) \quad \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A + \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left(\rho(L^2+\rho^2)^2\partial_\rho A\right) \\
&\qquad - \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\,A - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2 A_\rho\right] \\
&\qquad\qquad - i\frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0
\end{aligned}
\tag{5.29}
$$

It is worth mentioning that eq. (1) in (5.9) contains the operator $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i)$ which acts differently on scalars and spherical vectors. Therefore, when this operator is applied on $\Phi = \varphi + \frac{L}{R^2}(\hat{\gamma}_2 A_3 - \hat{\gamma}_3 A_2)$, in principle one should split it as acting on φ and A_i separately. However, since in the present case $A_i = \nabla_i A$, exploiting the algebra of covariant derivatives and the properties of scalar harmonics in (5.28), it is easy to show that

$$
\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i\nabla_j A) = \nabla_i\nabla^i\nabla_j A - 2\nabla_j A = -l(l+2)\nabla_j A \tag{5.30}
$$

This is exactly the same relation satisfied by the scalar φ, so we are led to $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i\Phi) = -l(l+2)\Phi$. This confirms that considering Φ as an elementary scalar fluctuation is a consistent procedure.

Equations (5.29) are four equations for three unknowns $X_{III}, Y_{III}, Z_{III}$ and lead to non-trivial solutions only if they are compatible. Indeed it turns out that equation (4) is identically satisfied once the others are. We then concentrate on the first three equations.

We first solve equation (1). By observing that it is identical to the equation for the scalar χ (see eq. (5.12)) we immediately obtain

$$
X_{III}(\rho) = \rho^l(L^2+\rho^2)^{-n-l-1}F(-(n+l+1), -n; l+2; -\rho^2/L^2) \tag{5.31}
$$

where the quantization condition (5.17) has been used. As a consequence, the mass spectrum is

$$
M_\Phi(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \tag{5.32}
$$

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where $n \geq 0$, $l \geq 1$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$.

Equation (2) can be used to express the mode A in terms of Φ and A_ρ. Inserting the expressions (5.28) we obtain

$$Z_{III} = \frac{1}{l(l+2)} \left[\frac{1}{\rho} \partial_\rho (\rho^3 Y_{III}) + i \frac{L R^2 \rho^2}{(L^2+\rho^2)^2} (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2) X_{III} \right] \tag{5.33}$$

We then consider equation (3) which exhibits an actual coupling between X_{III} and Y_{III}. In order to solve for Y_{III} given the solution (5.31) for X_{III} we set

$$Y_{III}(\varrho) = \varrho^{l-1} (1+\varrho^2)^{-\alpha} \, P(\varrho) \tag{5.34}$$

Using the definitions (5.14) together with the quantization condition (5.17) and defining $y \equiv -\varrho^2$, after some algebra the equation for P reads

$$\begin{aligned} y(1-y)P''(y) + [(l+2) + (2n+l)\, y] \, P'(y) - n(n+l+1)P(y) \\ = \eta \, \frac{(1+y)}{(1-y)^2} F(-(n+l+1), -n; l+2; y) \end{aligned} \tag{5.35}$$

where we have defined $\eta \equiv i \frac{R^2}{2L^2} (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)$. This is an inhomogeneous hypergeometric equation whose source is a polynomial of degree n, solution of the corresponding homogeneous equation. The most general solution is then of the form

$$P(y) = c \, F(-(n+l+1), -n; l+2; y) + \bar{P}(y) \tag{5.36}$$

for arbitrary constant c, where $\bar{P}$ is a particular solution of (5.35). Exploiting the general identity

$$\begin{aligned} (1-y) \, F'(-(n+l+2), -n; l+1; y) \; + \; (n+l+2) \, F(-(n+l+2), -n; l+1; y) \\ = \frac{(n+l+1)(n+l+2)}{(l+1)} \, F(-(n+l+1), -n; l+2; y) \end{aligned} \tag{5.37}$$

valid for hypergeometric functions with integer coefficients, it is easy to show that a solution is given by

$$\bar{P}(y) = \eta \, \frac{(l+1)}{(n+l+1)(n+l+2)} \, \frac{F(-(n+l+2), -n; l+1; y)}{1-y} \tag{5.38}$$

The general solution of equation (3) is then

$$\begin{aligned} Y_{III}(\rho) \; = \; \rho^{l-1} (L^2+\rho^2)^{-n-l-2} \Big[c \, (L^2+\rho^2) \, F(-(n+l+1), -n; l+2; -\rho^2/L^2) \\ + \eta \, \frac{(l+1)}{(n+l+1)(n+l+2)} F(-(n+l+2), -n; l+1; -\rho^2/L^2) \Big] \end{aligned} \tag{5.39}$$

This solution is regular at the origin and not divergent for $\rho \to \infty$. Due to the quantization condition (5.17) the corresponding mass spectrum is still given by

$$M_{III}(n, l, m_2, m_3) = \frac{2L}{R^2} \sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2} \right)^2} \tag{5.40}$$

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with $n \geq 0$, $l \geq 1$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$.

Before closing this section we comment on the particular $l = m_2 = m_3 = 0$ mode. In (5.28) this corresponds to turn off $A_i = \nabla_i A$ since $A(\sigma^a)$ is independent of the three-sphere coordinates. Equation (2) reduces to $\partial_\rho(\rho^3 A_\rho) = 0$ which, together with the condition of regularity at $\rho = 0$, sets $A_\rho = 0$. Equations (3) and (4) in (5.29) are then automatically satisfied, whereas eq. (1) provides a non-trivial solution for Φ as given in (5.31) with mass (5.32) where we set $l = m_2 = m_3 = 0$.

As a slightly different attitude we can consider the configuration with all the vector modes turned off ($Y_{III} = Z_{III} = 0$) and study only scalar Φ fluctuations of the form (5.28). In this case Φ is still solution of equation (1) but, as follows from the rest of equations, it is constrained by the further condition

$$(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\Phi = 0 \tag{5.41}$$

In general, for non-vanishing and distinct deformation parameters, non-trivial solutions can be found only for $m_2 = m_3 = 0$, i.e. only the $\mathrm{U}(1) \times \mathrm{U}(1)$ zero-mode sector is selected and the fluctuations are independent of (ϕ_2, ϕ_3). A greater number of solutions, corresponding to the modes $m_2 = m_3$, is instead allowed when $\hat{\gamma}_2 = \hat{\gamma}_3$, therefore in particular for the supersymmetric deformation. In any case, the mass spectrum is given by

$$M_\Phi(n,l) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2)} \qquad n \geq 0 \quad l \text{ (even)} \geq 0 \tag{5.42}$$

and coincides with the undeformed mass.

6. Analysis of the spectrum

From the previous discussion it follows that the bosonic modes arising from the compactification of the D7-brane on the deformed $\tilde{\mathrm{S}}^3$ give rise to a mesonic spectrum which is given by

- 2 scalars and 1 vector in the $(\frac{l}{2}, \frac{l}{2})$ with $l \geq 0$, $|m_2 \pm m_3| = l - 2k$ and mass

$$M_{\chi,\Phi,II}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

- 1 scalar in the $(\frac{l}{2}, \frac{l}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l - 2k$ and mass

$$M_{III}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

- 1 scalar in the $(\frac{l-1}{2}, \frac{l+1}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l \mp 1 - 2k$ and mass

$$M_{I,+}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+2)(n+l+3) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

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- 1 scalar in the $(\frac{l+1}{2}, \frac{l-1}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l \pm 1 - 2k$ and mass

$$M_{I,-}(n, l, m_2, m_3) = \frac{2L}{R^2}\sqrt{(n+l)(n+l+1) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

for any $n \geq 0$. This matches exactly the bosonic content found in the undeformed case [5]. However, in this case the γ-deformation breaks $\mathrm{SO}(4) \to \mathrm{U}(1) \times \mathrm{U}(1)$ and induces an explicit dependence of the mass spectrum on the the quantum numbers (m_2, m_3) with a pattern similar to the Zeeman effect for atomic electrons where the constant magnetic field which breaks $\mathrm{SU}(2)$ rotational invariance down to $\mathrm{U}(1)$ induces a dependence of the energy levels on the azimuthal quantum number m.[6]

The dependence on the deformation parameters disappears completely in the $m_2 = m_3 = 0$ sector (or for $\hat{\gamma}_2 = \hat{\gamma}_3$ and $m_2 = m_3$) and the mass eigenvalues coincide with the ones of the undeformed theory. When $\hat{\gamma}_2 = \hat{\gamma}_3$ the mass spectrum acquires an extra symmetry under the exchange of the two $\mathrm{U}(1)$'s and an extra degeneracy corresponding to $m_2 \to m_2 + m$, $m_3 \to m_3 + m$, m integer.

For any value of $\hat{\gamma}_i$ there are no tachyonic modes, so confirming the stability of our configuration. Moreover, massless states are absent and the spectrum has a mass gap given by

$$M_{gap} = 2\sqrt{2}\frac{L}{R^2} \tag{6.1}$$

This is exactly the mass gap present in the undeformed theory [5].

In order to analyze in detail the mass splitting induced by the deformation and study how the modes organize themselves among the different eigenvalues it is convenient to rewrite the mass of a generic eigenstate X as

$$M_X(n, l, m_2, m_3) = \sqrt{\left(M_X^{(0)}(n, l)\right)^2 + \frac{4L^2}{R^4}\left(\Delta M(m_2, m_3)\right)^2} \tag{6.2}$$

where $M_X^{(0)}$ is the undeformed mass, whereas

$$\Delta M(m_2, m_3) \equiv \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right) \tag{6.3}$$

is the Zeeman-splitting term.

Since for any $l \geq 2$ the following mass degeneracy occurs

$$M^{(0)}_{\chi,\Phi,II}(n, l) = M^{(0)}_{III}(n, l) = M^{(0)}_{I,+}(n, l-1) = M^{(0)}_{I,-}(n, l+1) \tag{6.4}$$

for $\hat{\gamma}_i = 0$ we have $8(l+1)^2$ bosonic degrees of freedom corresponding to the same mass eigenvalue. For the particular values $l = 0, 1$ the number of states is reduced since for $l = 0$ modes $A_{(I,+)}$ and A_{III} are both absent, whereas for $l = 1$ $A_{(I,+)}$ is still absent. For any value of l they match the bosonic content of massive $\mathcal{N} = 2$ supermultiplets [5].

[6] A similar effect has been observed in the case of backgrounds with B fields turned on in Minkowski [18, 35].

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State	$\lvert m_2 - m_3\rvert = 2j$	Degeneracy
χ, Φ, A_{III}	0	$l+1$
	$2, 4, \cdots, l$	$2(l+1)$
A_μ	0	$l+1$
	$2, 4, \cdots, l$	$2(l+1)$
$A_{I,+}$	0	$l-1$
	$2, 4, \cdots, l$	$2(l-1)$
$A_{I,-}$	0	$l+3$
	$2, 4, \cdots, l$	$2(l+3)$

Table 1: Degeneracy of states in the case $\hat{\gamma}_2 = \hat{\gamma}_3$ and $l \geq 2$ even. The degeneracy in the third column refers to every single value of j.

State	$\lvert m_2 - m_3\rvert = 2j+1$	Degeneracy
χ, Φ, A_{III}	$1, 3, \cdots, l$	$2(l+1)$
A_μ	$1, 3, \cdots, l$	$2(l+1)$
$A_{I,+}$	$1, 3, \cdots, l$	$2(l-1)$
$A_{I,-}$	$1, 3, \cdots, l$	$2(l+3)$

Table 2: Degeneracy of states in the case $\hat{\gamma}_2 = \hat{\gamma}_3$ and $l \geq 3$ odd.

In the present case mass degeneracy occurs among states which satisfy the above condition and have the same value of $\Delta M(m_2, m_3)$. Therefore, having performed the l-shift for the $(I, \pm)$ modes as in (6.4), we concentrate on the degeneracy in $\Delta M(m_2, m_3)$ for fixed values of (n, l). It is convenient to discuss the $\hat{\gamma}_2 = \hat{\gamma}_3$ and $\hat{\gamma}_2 \neq \hat{\gamma}_3$ cases, separately.

$\boldsymbol{\hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma}}$**:** this case includes the supersymmetric LM-theory. The deformation enters the mass spectrum only through the difference $(m_2 - m_3)$ and the splitting term ΔM depends only on a single integer j

$$\begin{aligned} l \quad \text{even} \qquad & 2j \equiv |m_2 - m_3| = 0, 2, \cdots, l \qquad & \Delta M(j) = \hat{\gamma}\, j \\ l \quad \text{odd} \qquad & 2j+1 \equiv |m_2 - m_3| = 1, 3, \cdots, l \qquad & \Delta M(j) = \hat{\gamma}\left(j + \frac{1}{2}\right) \end{aligned} \tag{6.5}$$

Excluding for the moment the $l = 0, 1$ cases, for any given value of $2j$ and $2j+1$ the degeneracies of the corresponding mass levels are listed in table 1 and table 2, respectively.

For any value of $l \geq 2$ we observe Zeeman-like splitting as shown in figure 1. Precisely, the splitting occurs in the following way: For l even there are $8(l+1)$ d.o.f. corresponding to $j = 0$ and $16(l+1)$ for each $j \neq 0$. Since we have $l/2$ possible values of $j \neq 0$, the total number of states sum up correctly to $8(l+1)^2$. Analogously, for odd values of l the number of levels is $(l+1)/2$, each of them corresponds to $16(l+1)$ d.o.f., so we still have $8(l+1)^2$ modes.

The $l = 0$ case corresponds to $m_2 = m_3 = 0$ $(j = 0)$. The deformation is then harmless and we are back to the bosonic content of the undeformed theory, that is three scalars χ, Φ,

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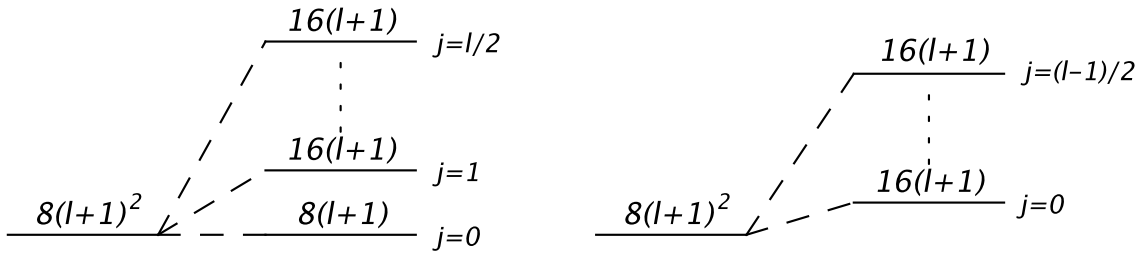

Figure 1: The Zeeman-splitting of the undeformed $8(l+1)^2$ d.o.f. for $\hat{\gamma}_2 = \hat{\gamma}_3$ and l even (left) or odd (right).

$A_{(I,-)}$ and one vector with $M^{(0)}(n,0)$. Similarly, for $l = 1$ ($j = 0$), excluding $A_{(I,+)}$ we have three scalars and one vector in the $(1/2, 1/2)$ of SO(4) and one scalar in the $(3/2, 1/2)$, all corresponding to $M^2 = (M^{(0)}(n,1))^2 + \hat{\gamma}^2 L^2/R^4$. These cases can be included in tables 1 and 2 with the agreement to discharge modes which are not switched on.

We note that there is an accidental mass degeneracy which is remnant of the undeformed $\mathcal{N} = 2$ theory. In particular, in the supersymmetric LM case this allows to organize the bosonic states in $\mathcal{N} = 1$ supermultiplets.

In principle, this unexpected degeneracy could be related to the particular theories we are considering which are smooth deformations of their undeformed counterpart. In order to better understand $\mathcal{N} = 2$ vs. $\mathcal{N} = 1$ supersymmetry at the level of mesonic spectrum, the study of the fermionic sector is a mandatory requirement.

$\hat{\gamma}_2 \neq \hat{\gamma}_3$: the splitting term ΔM now depends on both $m_{2,3}$ and no longer on their difference. In order to make the comparison with the $\hat{\gamma}_2 = \hat{\gamma}_3$ case easier, for fixed l it is convenient to label ΔM by two numbers j and s

$$\begin{aligned} l \quad \text{even} \qquad \Delta M(j,s) &= \frac{(j+s)\,\hat{\gamma}_2 + (j-s)\,\hat{\gamma}_3}{2} \\ l \quad \text{odd} \qquad \Delta M(j,s) &= \frac{(j+\frac{1}{2}+s)\,\hat{\gamma}_2 + (j+\frac{1}{2}-s)\,\hat{\gamma}_3}{2} \end{aligned} \tag{6.6}$$

where j is still defined as before, whereas s is integer if l is even and half-integer if l is odd. Its range can be read in tables 3 and 4.

As appears in the tables the degeneracy is almost completely broken. In fact, except for the $m_2 = m_3 = 0$ case, only a residual degeneracy 2 survives due to the fact that the mass (6.2) is invariant under the exchange $(m_2, m_3) \to (-m_2, -m_3)$.

To better understand the level splitting it is convenient to compare the present situation with the previous one. In fact, fixing j, the degenerate degrees of freedom of the $\hat{\gamma}_2 = \hat{\gamma}_3$ case further split according to the different values of s. If l is even and $j = 0$, the previous $8(l+1)$ degenerate levels split in $(l/2+2)$ new mass levels, while for $j \neq 0$ the $16(l+1)$ levels open up in $(l+3)$ levels (see figure 2). If l is odd we find $(l+3)$ different mass levels as drawn in figure 3.

The particular cases $l = 0, 1$ can be read from tables 3 and 4 by discharging $(A_{(I,+)}, A_{III})$ and $A_{(I,+)}$, respectively. For $l = 0$ three modes χ, Φ and A_μ correspond

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State	$\lvert m_2 - m_3\rvert = 2j$	s	Degeneracy
χ, Φ, A_{III}	0	0	1
		$1, 2, \cdots, \frac{l}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l}{2}, \cdots, 0, \cdots, \frac{l}{2}$	2
A_μ	0	0	1
		$1, 2, \cdots, \frac{l}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l}{2}, \cdots, 0, \cdots, \frac{l}{2}$	2
$A_{I,+}$	0	0	1
		$1, 2, \cdots, \frac{l-2}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l-2}{2}, \cdots, 0, \cdots, \frac{l-2}{2}$	2
$A_{I,-}$	0	0	1
		$1, 2, \cdots, \frac{l+2}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l+2}{2}, \cdots, 0, \cdots, \frac{l+2}{2}$	2

Table 3: Degeneracy of states in the case $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and $l \geq 2$ even. The degeneracy in the fourth column refers to every single pair (j, s).

State	$\lvert m_2 - m_3\rvert = 2j + 1$	s	Degeneracy
χ, Φ, A_{III}	$1, 3, \cdots, l$	$-\frac{l}{2}, \cdots, \frac{l}{2}$	2
A_μ	$1, 3, \cdots, l$	$-\frac{l}{2}, \cdots, \frac{l}{2}$	2
$A_{I,+}$	$1, 3, \cdots, l$	$-\frac{l-2}{2}, \cdots, \frac{l-2}{2}$	2
$A_{I,-}$	$1, 3, \cdots, l$	$-\frac{l+2}{2}, \cdots, \frac{l+2}{2}$	2

Table 4: Degeneracy of states in the case $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and $l \geq 3$ odd.

to $\Delta M = 0$ ($j = s = 0$), whereas the three degrees of freedom of $A_{(I,-)}$ split into one d.o.f. with $\Delta M = 0$ ($j = s = 0$) and two with $\Delta M = \frac{\hat{\gamma}_2 - \hat{\gamma}_3}{2}$ ($j = s = 1$). Already in the simplest $l = 0$ case the SO(4) breaking is manifest. For $l = 1$ ($j = 0$) the four degrees of freedom of each mode χ, Φ, A_{III} and A_μ now split into two states with $\Delta M = \hat{\gamma}_2/2$ and two states with $\Delta M = \hat{\gamma}_3/2$. On the other hand, the 8 d.o.f. corresponding to $A_{(I,-)}$ split into two states with $\Delta M = \hat{\gamma}_2/2$, two states with $\Delta M = \hat{\gamma}_3/2$, two states with $\Delta M = (2\hat{\gamma}_2 - \hat{\gamma}_3)/2$ and two with $\Delta M = (2\hat{\gamma}_3 - \hat{\gamma}_2)/2$.

As discussed in [5] the undeformed spectrum exhibits a huge degeneracy in $\nu \equiv n + l$ which can be traced back to a (non-exact) SO(5) symmetry. This originates from the fact that the induced metric on the D7-brane is conformally equivalent to $E^{(1,3)} \times \mathrm{S}^4$. If in the quadratic action for the fluctuations the conformal factor can be re-absorbed by a field redefinition the corresponding equations of motion are invariant under S^4 diffeomorphisms. Therefore, solutions can be found by expanding in spherical harmonics of S^4 and the mass spectrum of the elementary modes depends only on the SO(5) quantum number ν. This

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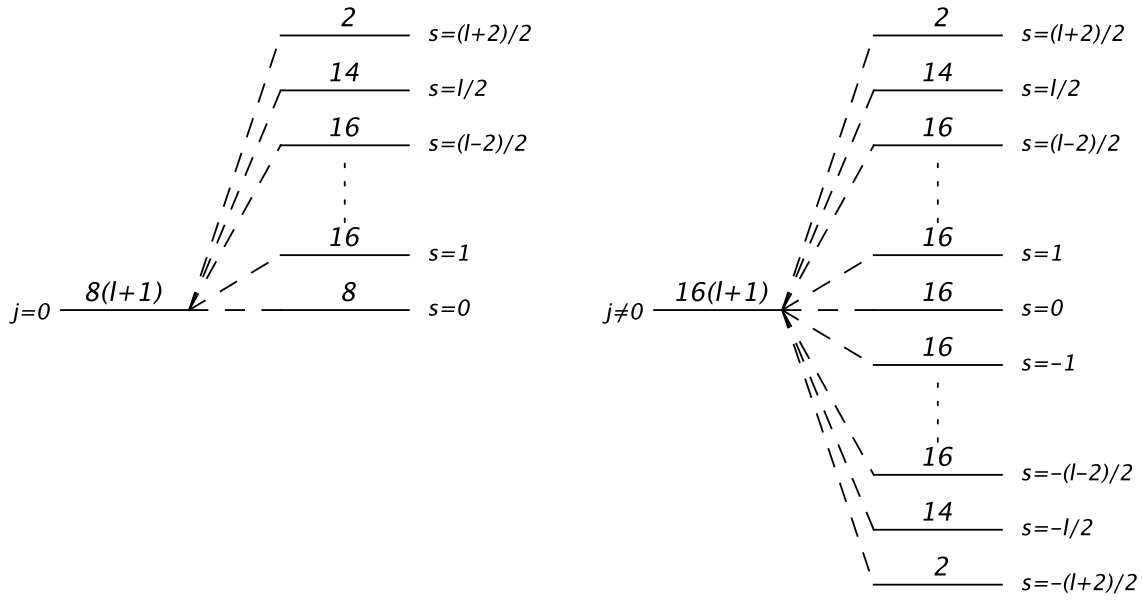

Figure 2: The Zeeman-splitting of the $\hat{\gamma}_2 = \hat{\gamma}_3 = \hat{\gamma}$ d.o.f. for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and l even. The value of ΔM here appearing is pictured considering the case $\hat{\gamma}_3 < \hat{\gamma} < \hat{\gamma}_2$.

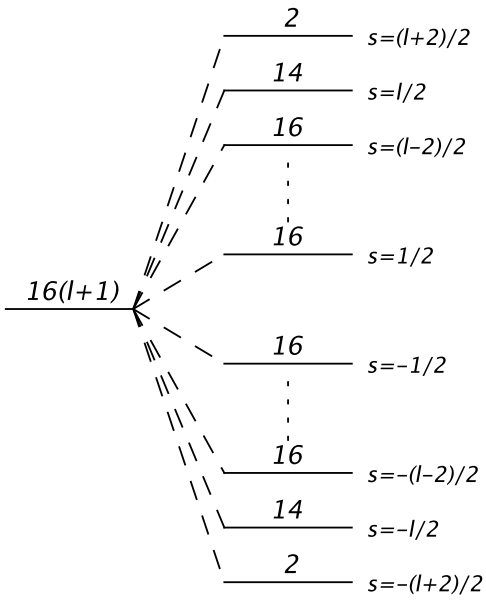

Figure 3: The Zeeman-splitting of the $\hat{\gamma}_2 = \hat{\gamma}_3$ d.o.f. for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and l odd. Once again $\hat{\gamma}_3 < \hat{\gamma} < \hat{\gamma}_2$.

happens for instance for scalar modes and vectors which, for a given ν, organize themselves into reducible representations $(0,0) \oplus (1/2,1/2) \cdots \oplus (\nu/2,\nu/2)$ of SO(4). This is indeed the decomposition of the highest weight representation $[\nu, 0]$ of SO(5) in SO(4) representations.

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In principle, the same analysis can be applied also to our case. Here the induced metric (3.11) is conformally equivalent to $E^{(1,3)} \times \tilde{\mathrm{S}}^4$ where $\tilde{\mathrm{S}}^4$ is the deformed four-sphere (set $\varrho = \rho/L$)

$$ds^2_{\tilde{\mathrm{S}}^4} = \frac{R^4}{4L^2}\frac{4}{(1+\varrho^2)^2}(d\varrho^2 + \varrho^2 d\tilde{\Omega}_3^2) \tag{6.7}$$

and

$$d\tilde{\Omega}_3^2 = d\theta^2 + G\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\varrho^2 c_\theta^2 s_\theta^2 (\hat{\gamma}_2 d\phi_2 + \hat{\gamma}_3 d\phi_3)^2}{(1+\varrho^2)^2}\right] \tag{6.8}$$

is the deformed three-sphere.

It follows that a dependence on the SO(5) quantum number $\nu = n + l$ still appears if the conformal factor $(1+\varrho^2)L^2/R^2$ can be compensated by a field redefinition and the action can be entirely expressed in terms of the metric of $E^{(1,3)} \times \mathrm{S}^4$ plus deformations. A close look at the action (4.13) reveals that this is always the case for the decoupled modes χ, A_μ and also for Φ. Despite of the presence of the deformation terms which break explicitly the SO(5) invariance, we can still search for solutions expanded in spherical harmonics on S^4 and, consequently, the mass spectrum exhibits a dependence on n and l only in the combination $n + l$. In particular, in the zero-mode sector $m_2 = m_3 = 0$ a degeneracy appears which is remnant of the SO(5) invariance. Of course, the eigenstates corresponding to degenerate eigenvalues never reconstruct the complete $[\nu, 0]$ representation of SO(5), being organized into a direct product of SO(4) representations with integer spins only $(0,0) \oplus (1,1) \cdots ([\nu/2], [\nu/2])$, since $m_2 = m_3 = 0$ only occurs for even values of l.

7. The dual field theory

In this section we construct the 4D conformal field theory whose composite operators are dual to the mesonic states just found.

As already discussed in section 3, in the supergravity description the operations of TsT deforming the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and adding D7-branes commute. Since on the field theory side TsT deformations correspond to promoting all the products among the fields to be $*$-products [21], whereas the addition of D7-branes corresponds to adding interacting fundamental matter [2] we expect that in determining the action for the dual field theory the operations of $*$-product deformation and addition of fundamental matter commute. Therefore, in order to obtain the dual action we proceed by promoting to $*$-products all the products in the $\mathcal{N} = 2$ SYM action with fundamental matter corresponding to the undeformed Karch-Katz model.

Given N_f probe D7-branes embedded in the ordinary $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with N units of flux, $N \gg N_f$, in the large N limit the dual field theory on the D3-branes consists of $\mathcal{N} = 4$ SU(N) SYM coupled in a $\mathcal{N} = 2$ fashion to N_f $\mathcal{N} = 2$ hypermultiplets which contain new dynamical fields arising from open strings stretching between D3 and D7-branes. In $\mathcal{N} = 1$ superspace language the $\mathcal{N} = 4$ gauge multiplet is given in terms of one $\mathcal{N} = 1$ gauge superfield W_α and three chirals Φ_1, Φ_2, Φ_3 all in the adjoint representation of SU(N). The $\mathcal{N} = 2$ hypermultiplets are described by N_f chiral superfields Q^r transforming in the $(N, \bar{N}_f)$ of SU(N) $\times$ SU(N_f) plus N_f chirals $\tilde{Q}_r$ transforming in the $(\bar{N}, N_f)$.

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According to the AdS/CFT duality the lowest components of the three chirals Φ_i are in one-to-one correspondence with the three complex coordinates of the internal 6D space as (we use notations consistent with section 2)

$$\begin{aligned} X^1 + iX^2 &\equiv u\rho_3 e^{i\phi_3} \to \Phi_3|_{\theta=\bar\theta=0} \\ X^3 + iX^4 &\equiv u\rho_2 e^{i\phi_2} \to \Phi_2|_{\theta=\bar\theta=0} \\ X^5 + iX^6 &\equiv u\rho_1 e^{i\phi_1} \to \Phi_1|_{\theta=\bar\theta=0} \end{aligned} \tag{7.1}$$

For a configuration of D7-branes placed at distance $X^5 + iX^6 = L$ from the D3-branes the Lagrangian of the corresponding gauge theory is [2]

$$\begin{aligned} \mathcal{L} = &\int d^4\theta \left[\mathrm{Tr}\left(e^{-gV}\bar\Phi_i e^{gV}\Phi^i\right) + \mathrm{tr}\left(\bar Q e^{gV} Q + \tilde Q e^{-gV}\bar{\tilde Q}\right)\right] + \frac{1}{2g^2}\int d^2\theta\, \mathrm{Tr}\left(W^\alpha W_\alpha\right) \\ &+ i\int d^2\theta \left[g\,\mathrm{Tr}\left(\Phi^1\left[\Phi^2,\Phi^3\right]\right) + g\,\mathrm{tr}\left(\tilde Q\Phi^1 Q\right) + m\,\mathrm{tr}\left(\tilde Q Q\right)\right] + h.c. \end{aligned} \tag{7.2}$$

where the trace Tr is over color indices and tr is over the flavor ones. This action is $\mathcal{N}=2$ supersymmetric with (W_α, Φ_1) realizing a $\mathcal{N}=2$ vector multiplet and (Φ_2, Φ_3) an adjoint matter hypermultiplet. The coupling of Φ_1 with massive matter fields leads to a non-trivial vev $\langle\Phi_1\rangle = -m/g$ which gives the displacement between the D3 and the D7-branes according to the identification $L \equiv -m/g$.

The theory has a $\mathrm{SU}(2)_\Phi \times \mathrm{SU}(2)_R$ invariance corresponding to a symmetry which exchanges (Φ_2, Φ_3) and to the $\mathcal{N}=2$ R-symmetry, respectively. In addition, for $m=0$, there is a $\mathrm{U}(1)$ R-symmetry under which $(Q^r, \tilde Q_r)$ and (Φ_2, Φ_3) are neutral, whereas Φ_1 has charge 2 and W_α has charge 1 [36, 16]. In the dual supergravity description these symmetries originate from the $\mathrm{SO}(4)\times\mathrm{SO}(2)$ invariance which survives after the insertion of the D7-branes [2] and which are related to rotations in the (X^1, X^2, X^3, X^4) and (X^5, X^6) planes, respectively. Fixing $X^5 + iX^6 = L \neq 0$ breaks rotational invariance in the (X^5, X^6) plane and, correspondingly, the mass term breaks the $\mathrm{U}(1)$ R-symmetry in the dual gauge theory. Finally, the theory also possesses a $\mathrm{U}(1)$ baryonic symmetry under which only $(Q^r, \tilde Q_r)$ are charged $(1, -1)$. This is a residual of the original $\mathrm{U}(N_f)$ invariance.

For $m=0$ and in the large N limit with N_f fixed the theory is superconformal invariant. In fact, the beta-function for the 't Hooft coupling $\lambda = g^2 N$ is proportional to $\lambda^2 N_f/N$ and vanishes for $N_f/N \to 0$.

Since we are interested in non-supersymmetric deformations of this theory we need the Lagrangian (7.2) expanded in components. Given the physical components of the multiplets being

$$\begin{aligned} \Phi^i &= \left(a^i, \psi^i_\alpha\right) & Q^r &= \left(q^r, \chi^r_\alpha\right) \\ W_\alpha &= \left(\lambda_\alpha, f_{\alpha\beta}\right) & \tilde Q_r &= \left(\tilde q_r, \tilde\chi_{r\alpha}\right) \end{aligned} \tag{7.3}$$

after eliminating the auxiliary fields through their algebraic equations of motion, the Lagrangian (7.2) takes the form

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\mathrm{int}} \tag{7.4}$$

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where[7]

$$\mathcal{L}_{\mathcal{N}=4} = \mathrm{Tr}\left(-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta} + i\lambda\left[\nabla,\bar{\lambda}\right] + \bar{a}_i\Box a^i + i\psi^i\left[\nabla,\bar{\psi}_i\right]\right)$$
$$+g^2\,\mathrm{Tr}\left(-\frac{1}{4}\left[a^i,\bar{a}_i\right]\left[a^j,\bar{a}_j\right] + \frac{1}{2}\left[a^i,a^j\right]\left[\bar{a}_i,\bar{a}_j\right]\right)$$
$$+\left\{ig\,\mathrm{Tr}\left(\left[\bar{\psi}_i,\bar{\lambda}\right]a^i + \frac{1}{2}\epsilon_{ijk}\left[\psi^i,\psi^j\right]a^k\right) \quad + h.c.\right\} \tag{7.5}$$

is the ordinary $\mathcal{N}=4$ Lagrangian,

$$\mathcal{L}_b = \mathrm{tr}\Big(\bar{q}\left(\Box - |m|^2\right)q + \tilde{q}\left(\Box - |m|^2\right)\bar{\tilde{q}}\Big)$$
$$-\frac{g^2}{4}\,\mathrm{tr}\Big(\bar{q}\,q\,\bar{q}\,q + \tilde{q}\,\bar{\tilde{q}}\,\tilde{q}\,\bar{\tilde{q}} - 2\bar{q}\,\bar{\tilde{q}}\,\tilde{q}\,q + 4\tilde{q}\,\bar{\tilde{q}}\,\bar{q}\,q\Big) + \frac{g^2}{2}\,\mathrm{tr}\left(\tilde{q}\left[a^i,\bar{a}_i\right]\bar{\tilde{q}} - \bar{q}\left[a^i,\bar{a}_i\right]q\right)$$
$$-\left\{\mathrm{tr}\left(g\bar{m}(\bar{q}a_1q + \tilde{q}a_1\bar{\tilde{q}}) + \frac{g^2}{2}\left(\bar{q}\bar{a}_1a^1q + \tilde{q}a^1\bar{a}_1\bar{\tilde{q}} + 2\tilde{q}\left[\bar{a}_2,\bar{a}_3\right]q\right)\right) \quad + h.c.\right\} \tag{7.6}$$

describes the bosonic fundamental sector and its interactions with bosonic matter in the adjoint,

$$\mathcal{L}_f = i\,\mathrm{tr}\left(\bar{\chi}\overrightarrow{\nabla}\chi - \tilde{\chi}\overleftarrow{\nabla}\bar{\tilde{\chi}}\right) + \left\{im\,\mathrm{tr}\left(\tilde{\chi}\chi\right) \quad + h.c.\right\} \tag{7.7}$$

describes the free fermionic fundamental sector and

$$\mathcal{L}_{\mathrm{int}} = ig\,\mathrm{tr}\Big(\bar{\chi}\bar{\lambda}q - \tilde{q}\bar{\lambda}\bar{\tilde{\chi}} + \tilde{q}\psi^1\chi + \tilde{\chi}\psi^1q + \tilde{\chi}a^1\chi\Big) \quad + h.c. \tag{7.8}$$

contains the interaction terms between bosons and fermions.

The most general non-supersymmetric marginal deformation of this theory can be obtained by promoting all the products among the fields in the Lagrangian to be $*$-products according to the following prescription [38]

$$f\,g \;\longrightarrow\; f*g = e^{i\pi Q_i^f Q_j^g \epsilon_{ijk}\gamma_k}\,f\,g \tag{7.9}$$

where γ_k are the deformation parameters, whereas (Q_1,Q_2,Q_3) are the charges of the fields under the three U(1) global symmetries of the original $\mathcal{N}=4$ theory associated to the Cartan generators of SU(4). On the dual supergravity side they correspond to angular shifts in (7.1). Accordingly, the charges of the chiral Φ_i superfields are chosen as in table 5 [38] with the additional requirement for the charges of the spinorial superspace coordinates to be $(1/2,1/2,1/2)$. This insures invariance of the superpotential term $\int d^2\theta\mathrm{Tr}(\Phi^1[\Phi^2,\Phi^3])$ under the three U(1)'s. The charges for the matter chiral superfields are determined by requiring the superpotential term $\int d^2\theta\mathrm{tr}(\tilde{Q}\Phi^1Q)$ to respect the three global symmetries in addition to the condition for Q and $\tilde{Q}$ to have the same charges.

[7] We use superspace conventions of [37]. When $\psi\lambda$ indicates the product of two chiral fermions it has to be understood as $\psi^\alpha\lambda_\alpha$. The same convention is used for antichiral fermions.

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	Φ^1	Φ^2	Φ^3	Q	$\tilde{Q}$
Q_1	1	0	0	0	0
Q_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$
Q_3	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$

Table 5: U(1) charges of the chiral superfields. The corresponding antichirals have opposite charges.

The gauge superfield W_α and the gaugino have charges $(1/2, 1/2, 1/2)$, whereas the gauge field strength $f_{\alpha\beta}$ is neutral under the three U(1)'s.

In the absence of mass term in (7.2) the corresponding currents $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3})$ are conserved, whereas J_{ϕ_1} fails to be conserved when $m \neq 0$. Moreover, (J_{ϕ_2}, J_{ϕ_3}) are ABJ-anomaly free also in the presence of fundamental matter, whereas J_{ϕ_1} is non-anomalous only in the quenching limit $N_f/N \to 0$.

As is well-known, the ordinary Lunin-Maldacena U(1)×U(1) charges [21] are associated to (φ_1, φ_2) angular shifts after performing the change of variables (in our notations)

$$\varphi_1 = \frac{1}{3}(\phi_1 + \phi_2 - 2\phi_3), \quad \varphi_2 = \frac{1}{3}(\phi_2 + \phi_3 - 2\phi_1), \quad \varphi_3 = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3), \tag{7.10}$$

Expressing the $(J_{\varphi_1}, J_{\varphi_2})$ generators in terms of $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3})$ we easily find that the Lunin-Maldacena charges are given by

$$Q_1^{(LM)} = Q_2 - Q_3 \qquad , \qquad Q_2^{(LM)} = Q_2 - Q_1 \tag{7.11}$$

In the case of supersymmetric deformations the third linear combination $Q_R \sim (Q_1 + Q_2 + Q_3)$ provides the R-symmetry charge.

We are now ready to derive the deformed action by using the prescription (7.9) in the original undeformed one.

We begin with the one-parameter deformation, $\gamma_1 = \gamma_2 = \gamma_3$. In this case $\mathcal{N} = 1$ supersymmetry survives and we can work directly with the superspace action (7.2). Since only for $m = 0$ the $*$-product is well-defined being the three U(1) charges conserved, the correct way to proceed is to deform the massless theory and then add the mass operator as a perturbation. Following this prescription and taking into account the superfields charges given in table 5, the Lagrangian of the deformed theory is

$$\begin{aligned}\mathcal{L} = &\int d^4\theta \left[\mathrm{Tr}\left(e^{-g\,V}\bar{\Phi}_i e^{g\,V}\Phi^i\right) + \mathrm{tr}\left(\bar{Q}e^{g\,V}Q + \tilde{Q}e^{-g\,V}\bar{\tilde{Q}}\right)\right] + \frac{1}{2g^2}\int d^2\theta\,\mathrm{Tr}\left(W^\alpha W_\alpha\right) \\ &+ ig\int d^2\theta\,\left[\mathrm{Tr}\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right) + \mathrm{tr}\left(\tilde{Q}\Phi_1 Q\right) + m\,\mathrm{tr}\left(\tilde{Q}Q\right)\right]\end{aligned} \tag{7.12}$$

We note that a non-trivial deformation appears in the superpotential only in the pure adjoint sector. The interaction and the mass terms involving flavor matter do not change, so that the vev for Φ_1 which is related to the D7-brane location through the dictionary (7.1) is the same as in the undeformed theory, $\langle\Phi_1\rangle = -m/g \equiv L$. Since in the supergravity

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description we have chosen L to be real ($X^5 = L$, $X^6 = 0$) here and in what follows we restrict to real values of m.

As already stressed, for $m \neq 0$ the Q_1 charge is not conserved, neither is $Q_2^{(LM)}$. Therefore, this deformed theory possesses only one U(1) non-R–symmetry corresponding to $Q_1^{(LM)}$.

The action (7.12) has been obtained by $*$-product deforming the $\mathcal{N} = 2$ SYM action (7.2). However, it could have been equivalently obtained by adding fundamental chiral matter to the $\mathcal{N} = 1$ β-deformed SYM theory of [21]. In particular, the appearance of the gauge coupling constant in front of the adjoint chiral superpotential insures that for $m = 0$ and in the probe approximation the theory is superconformal invariant [39].

We now consider the more general non-supersymmetric case. We implement the $*$-product (7.9) in the action (7.4). Using the deformed commutator [38]

$$[X_i, X_j]_{M_{ij}} \equiv e^{i\pi M_{ij}} X_i X_j - e^{-i\pi M_{ij}} X_j X_i \tag{7.13}$$

where for X_i fermions

$$M_{\text{fermions}} \equiv B = \begin{pmatrix} 0 & \frac{1}{2}(\gamma_1+\gamma_2) & -\frac{1}{2}(\gamma_1+\gamma_3) & -\frac{1}{2}(\gamma_2-\gamma_3) \\ -\frac{1}{2}(\gamma_1+\gamma_2) & 0 & \frac{1}{2}(\gamma_2+\gamma_3) & -\frac{1}{2}(\gamma_3-\gamma_1) \\ \frac{1}{2}(\gamma_3+\gamma_1) & -\frac{1}{2}(\gamma_2+\gamma_3) & 0 & -\frac{1}{2}(\gamma_1-\gamma_2) \\ \frac{1}{2}(\gamma_2-\gamma_3) & \frac{1}{2}(\gamma_3-\gamma_1) & \frac{1}{2}(\gamma_1-\gamma_2) & 0 \end{pmatrix} \tag{7.14}$$

whereas for scalars

$$M_{\text{scalars}} \equiv C = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix} \tag{7.15}$$

the deformed $\mathcal{L}_{\mathcal{N}=4}$ takes the form

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} = \operatorname{Tr}&\left(-\frac{1}{2} f^{\alpha\beta} f_{\alpha\beta} + i\lambda\left[\nabla, \bar{\lambda}\right] + \bar{a}_i \Box a^i + i\psi^i\left[\nabla, \bar{\psi}_i\right]\right) \\ &+ g^2 \operatorname{Tr}\left(-\frac{1}{4}\left[a^i, \bar{a}_i\right]\left[a^j, \bar{a}_j\right] + \frac{1}{2}\left[a^i, a^j\right]_{C_{ij}}\left[\bar{a}_i, \bar{a}_j\right]_{C_{ij}}\right) \\ &+ \left\{ ig \operatorname{Tr}\left(\left[\bar{\psi}_i, \bar{\lambda}\right]_{B_{i4}} a^i + \frac{1}{2}\epsilon_{ijk}\left[\psi^i, \psi^j\right]_{B_{ij}} a^k\right) \quad + h.c. \right\} \end{aligned} \tag{7.16}$$

while the bosonic sector reads

$$\begin{aligned} \mathcal{L}_b = \operatorname{tr}&\Big(\bar{q}\left(\Box - m^2\right) q + \tilde{q}\left(\Box - m^2\right)\bar{\tilde{q}}\Big) - \frac{g^2}{4}\operatorname{tr}\Big(\bar{q}\, q\, \bar{q}\, q + \tilde{q}\,\bar{\tilde{q}}\,\tilde{q}\,\bar{\tilde{q}} - 2\bar{q}\,\bar{\tilde{q}}\,\tilde{q}\, q + 4\tilde{q}\,\bar{\tilde{q}}\,\bar{q}\, q\Big) \\ &+ \frac{g^2}{2}\operatorname{tr}\left(\tilde{q}\left[a^i, \bar{a}_i\right]\bar{\tilde{q}} - \bar{q}\left[a^i, \bar{a}_i\right] q + \bar{q}\bar{a}_1 a^1 q + \tilde{q} a^1 \bar{a}_1 \bar{\tilde{q}}\right) \\ &+ \Big\{ g^2 \operatorname{tr}\left(\tilde{q}\left[\bar{a}_2, \bar{a}_3\right]_{C_{23}} q\right) - gm \operatorname{tr}\left(e^{-i\pi(\gamma_2-\gamma_3)}\bar{q} a^1 q + e^{i\pi(\gamma_2-\gamma_3)}\tilde{q} a^1 \bar{\tilde{q}}\right) \quad + h.c. \Big\} \end{aligned} \tag{7.17}$$

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and the fermionic one

$$\mathcal{L}_f = i\,\mathrm{tr}\left(\bar{\chi}\overrightarrow{\nabla}\chi - \tilde{\chi}\overleftarrow{\nabla}\bar{\tilde{\chi}}\right) + \left\{im\,\mathrm{tr}\left(\tilde{\chi}\chi\right) \quad + h.c.\right\} \tag{7.18}$$

Finally the boson-fermion interaction terms become

$$\begin{aligned}\mathcal{L}_{\mathrm{int}} = ig\,\mathrm{tr}\Big(& e^{i\frac{\pi}{4}(\gamma_2-\gamma_3)}\bar{\chi}\bar{\lambda}q - e^{-i\frac{\pi}{4}(\gamma_2-\gamma_3)}\tilde{q}\bar{\lambda}\bar{\tilde{\chi}} \\ & + e^{i\frac{\pi}{4}(\gamma_2-\gamma_3)}\tilde{q}\psi^1\chi + e^{-i\frac{\pi}{4}(\gamma_2-\gamma_3)}\tilde{\chi}\psi^1 q + \tilde{\chi}a^1\chi\Big) + h.c.\end{aligned} \tag{7.19}$$

We observe that the fundamental fields q and $\tilde{q}$ experiment the γ_1-deformation only through the modified commutator $[\bar{a}_2, \bar{a}_3]_{C_{23}}$ in $\mathcal{L}_b$. Moreover, γ_2 and γ_3 are always present in the combination $(\gamma_2 - \gamma_3)$ so that the corresponding phases disappear when $\gamma_2 = \gamma_3$, in particular for supersymmetric deformations.

8. Conclusions

In this paper we have studied the embedding of D7-branes in LM-Frolov backgrounds with the aim of finding the mesonic spectrum of the dual Yang-Mills theory with flavors, according to the gauge/gravity correspondence. Since these theories have $\mathcal{N} = 1$ or no supersymmetry depending on the choice of the deformation parameters $\hat{\gamma}_i$, they provide an interesting playground in the study of generalizations of the AdS/CFT correspondence to more realistic models with less supersymmetry.

These geometries are smoothly related to the standard $\mathrm{AdS}_5 \times \mathrm{S}^5$ from which they can be obtained by operating with TsT transformations. Therefore, if we consider D7-brane embeddings which closely mimic the ones of the undeformed case [2] we expect the flavor probes to share some properties with the probes of the undeformed case. Driven by this observation we have considered a spacetime filling D7-brane wrapped on a deformed three-sphere in the internal coordinates. We have found that for both the supersymmetric and the non-supersymmetric deformations a static configuration exists which is completely independent of the specific values of the deformation parameters $\hat{\gamma}_i$. As a consequence the D7-brane still lies at fixed values of its transverse directions and exhibits no quark condensate [2]. We remark that this shape is exact and stable in the supersymmetric as well as in the non-supersymmetric cases.

Although the shape of the brane does not feel the effects of the deformation, its fluctuations do. In fact, studying the scalar and vector fluctuations we have found that a non-trivial dependence on the $\hat{\gamma}_{2,3}$ parameters appears both in terms which correct the free dynamics of the modes and in terms which couple the U(1) worldvolume gauge field to one of the scalars in the mutual orthogonal directions to the D3-D7 system. All the deformation-dependent contributions arise from the Dirac-Born-Infeld term in the D7-brane action, whereas the Wess-Zumino term does not feel the deformation. The $\hat{\gamma}_1$ parameter, associated to a TsT transformation along the torus inside the D7 worldvolume, never enters the equations of motion.

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A smooth limit to the undeformed equations of motion exists for $\hat{\gamma}_i \to 0$. In this limit all the modes decouple and we are back to the undeformed solutions of [5]. The effect of the deformations becomes negligible also in the UV limit ($\rho \to \infty$). This is an expected result since the deformations involve tori in the internal space and in the UV limit the metric of the background reduces to flat four dimensional Minkowski spacetime.

On the other hand, the situation changes once we consider the general deformed equations. In fact, solving analytically these equations for elementary excitations of scalars and vectors we have found that the mass spectrum is still discrete and with a mass gap and the corresponding eigenstates match the one of the undeformed case. However, the mass eigenvalues acquire a non-trivial dependence on $\hat{\gamma}_{2,3}$. These new terms, being proportional to the $\mathrm{U}(1) \times \mathrm{U}(1)$ quantum numbers (m_2, m_3), induce a level spitting according to a Zeeman-like effect.

We have performed a detailed analysis of the level splitting and of the corresponding degeneracy. The situation turns out to be very different according to $\hat{\gamma}_2$ and $\hat{\gamma}_3$ being equal or not. In fact, for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ the degeneracy is almost completely broken since only a residual degeneracy associated to the invariance of the mass under $(m_2, m_3) \to (-m_2, -m_3)$ survives. In particular, the breaking of SO(4) is manifest. Instead, for $\hat{\gamma}_2 = \hat{\gamma}_3$ the mass levels split but for each value of the mass an accidental degeneracy survives which is remnant of the $\mathcal{N} = 2$ case. While in the supersymmetric case ($\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3$) this allows to arrange mesons in massive $\mathcal{N} = 1$ multiplets according to the fact that our embedding preserves supersymmetry, this higher degree of degeneracy in the bosonic sector of the theory does not have a clear explanation at the moment. In order to make definite statements about the supersymmetry properties of the mesonic spectrum and supersymmetry breaking one should study the fermionic sector. A useful strategy could be the bottom-up approach described in [16]. We leave this interesting open problem for the future.

Our analysis shares some similarities with other cases considered in the literature.

First of all, we have found that a stable embedding of the probe brane can be realized which is static and independent of the deformation parameters. This feature has been already encountered for other brane configurations in deformed backgrounds. An example is given by particular dynamical probe D3-branes (giant gravitons) which have been first well understood in [26]. In fact, there it has been shown that giant gravitons exist and are stable even in the absence of supersymmetry and their dynamics turns out to be completely independent of the deformation parameters, being then equal to the one of the undeformed theory. Moreover, since the giants wrap the same cycle inside the internal deformed space as our D7-brane does, their bosonic fluctuations encode the same dependence on the deformation parameters observed in the mesonic spectrum coming from the D7.

A second similarity emerges with the case of flavors in non-commutative theories investigated in [10]. In fact, the non-trivial coupling between scalar and gauge modes that in our case is induced by the deformation resembles the one which appears in the case of D7-branes embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with a B field turned on along spacetime directions. This is not surprising since both theories can be obtained performing a TsT transformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$: If the TsT is performed in AdS one obtains the dual of a non-commutative theory while the LM-Frolov picture is recovered if this transformation deforms the internal S^5.

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The field theory dual to the (super)gravity picture we have considered can be obtained by deforming the standard action for $\mathcal{N} = 4$ super Yang-Mills coupled to massive $\mathcal{N} = 2$ hypermultiplets by the $*$-product prescription [21]. In principle, in the supergravity dual description this should correspond to performing a TsT deformation *after* the embedding of the probe brane. However, as we have discussed, adding the flavor brane in the deformed background or deforming the Karch-Katz D3-D7 configuration are commuting operations. Therefore, the prescription we propose on the field theory side is consistent with what we have done on the string theory side. It is important to stress that the choice of the embedding we have made is crucial for the above reasoning.

What we obtain is a deformed gauge field theory with massive fundamental matter parametrized by four real parameters γ_i and m. We can play with them in order to break global $\mathrm{U}(1)$ symmetries, conformality and/or supersymmetry in a very controlled way. In fact, in the quenching approximation a non-vanishing mass parameter related to the location of the probe in the dual geometry breaks conformal invariance and one of the $\mathrm{U}(1)$ global symmetries of the massless theory. On the other hand, the values of the deformation parameters γ_i determine the degree of supersymmetry of the theory, as already discussed. It is interesting to note that as we found on the gravity side, the three deformation parameters play different roles in the fundamental sector of the theory. In fact, $\gamma_{2,3}$ always appear in the combination $(\gamma_2 - \gamma_3)$, so that if $\gamma_2 = \gamma_3$ this sector gets deformed only by γ_1-dependent phases induced by the interaction with the adjoint matter. In the supersymmetric case this particular behavior is manifest when using superspace formalism since a non-trivial deformation appears only in the adjoint sector, whereas the flavor superpotential remains undeformed.

Let us conclude mentioning some directions in which our work could be extended. We have considered only the non-interacting mesonic sector. Expanding the D7-brane action beyond the second order in α' one can get informations on the interactions among the mesons and understand how the deformation enters the couplings. Moreover, one could extend our analysis to mesons with large spin in Minkowski, similarly to what has been done in the ordinary, undeformed case [5].

Finally it could be very interesting to study in detail the other embeddings proposed in [28] and in particular the one which seems to exhibit chiral symmetry breaking. Moreover, going beyond the quenching approximation has been representing an interesting subject since the recent efforts to study back-reacted models [15].

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